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The influence of Lebesgue functions on the convergence and summability of function series

By G. ALEXITS in Budapest and A. SHARMA in Edmonton (Canada)

1. Introduction

Let X be a measurable space with a positive measure μ and let $\{f_n(x)\}$ be a sequence of μ-integrable functions on the measurable set $E \subset X$. Form the "Lebesgue functions"

$$L_n(x) = \int_E |K_n(t, x)|\, d\mu(t) \quad \text{and} \quad L_n^1(x) = \int_E |K_n^1(t, x)|\, d\mu(t),$$

where

$$K_n(t, x) = \sum_{k=0}^{n} f_k(t) f_k(x) \quad \text{and} \quad K_n^1(t, x) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) f_k(t) f_k(x).$$

These functions play an important role in the theory of convergence and summability of orthogonal series. We mention the following theorems of S. KACZMARZ [3], based on a method of A. KOLMOGOROFF—G. SELIVERSTOFF [4] and A. PLESSNER [5]:

A. If E is an interval of finite length on the real line, μ is the ordinary Lebesgue measure, and $\{f_n(x)\}$ is an orthonormal system defined on E, then the series $\Sigma a_n f_n(x)$ is convergent a.e. on E provided that $L_n(x) = O(\lambda_n)$ on E with $0 < \lambda_n \leqq \lambda_{n+1}$ and $\Sigma a_n^2 \lambda_n < \infty$.

B. Under the same conditions as above, $\Sigma a_n f_n(x)$ is $(C, 1)$-summable a.e. on E if, instead of $L_n(x) = O(\lambda_n)$, we only suppose $L_n^1(x) = O(\lambda_n)$ on E.

In the proof of these theorems the assumption that the system $\{f_n(x)\}$ is orthonormal was essentially exploited. Unexpectedly it turned out that *neither orthonormality nor L_μ^2-integrability of $\{f_n(x)\}$ is needed in theorems A and B.* It is enough to suppose that the functions $f_n(x)$ are L_μ-integrable on E and the condition

$$\int_E \left| \sum_{k=0}^{n} a_k b_k f_k(x) \right| d\mu(x) = O(1) \tag{1}$$

is satisfied whenever $\sum a_k^2 b_k^2 < \infty$.

We remark that (1) is trivially satisfied for orthonormal systems defined on a set E of finite measure, so that our results contain theorems A and B as special cases. Moreover, if we suppose $\lambda_n=1$ $(n=0, 1, \dots)$, i.e. if

$$L_n(x)=O(1) \quad \text{or} \quad L_n^1(x)=O(1) \tag{2}$$

on E, then even (1) is unnecessary. Hence *if one of the conditions* (2) *is uniformly valid on E, then $\Sigma a_n f_n(x)$ is a. e. convergent or $(C, 1)$-summable on E, respectively, under the sole condition $\Sigma a_n^2<\infty$*. So we can say that some classical theorems as for instance the theorem of Fejér—Lebesgue applied to the Fourier series of L^2-integrable functions is but a special case of our theorem belonging to the general theory of real functions.

As to the proof, we proceeded originally on the same way we followed in the case of multiplicatively orthogonal series (see [2]). C. I. PRESTON, after having read a preprint of [2], has communicated in a letter to the first author an idea which simplified also a part of our original proof very much. (The note of DR. PRESTON referring to this will appear later*). In the present paper we shall use his idea in the proof of Theorems 1 and 5.

2. The convergence problem

Let $\{f_n(x)\}$ be a sequence of L_μ-integrable functions on the μ-measurable set E and $\{\lambda_n\}$ a non-decreasing sequence of positive numbers. Denote further by $s_n(x)$ the n-th partial sum $\sum_{k=0}^{n} a_k f_k(x)$ of the series $\sum_{n=0}^{\infty} a_n f_n(x)$.

Theorem 1. *If $\Sigma a_n^2<\infty$ and the Lebesgue functions $L_{\nu_n}(x)$ satisfy the condition*

$$L_{\nu_n}(x)=O(\lambda_{\nu_n})$$

uniformly on the measurable set E of finite measure, then $s_{\nu_n}(x)=O_x(\lambda_{\nu_n}^{\frac{1}{2}})$ on E almost everywhere.

Denote by $n(x)$ the least index m $(\leqq n)$ for which

$$\lambda_{\nu_m}^{-\frac{1}{2}} s_{\nu_m}(x) = \max_{0\leqq k\leqq n} \lambda_{\nu_k}^{-\frac{1}{2}} s_{\nu_k}(x).$$

We proceed to prove that the left hand side is finite on E a. e. For this purpose we use an idea of Preston which consists in a special representation of $s_{\nu_{n(x)}}(x)$.

*) Meanwhile it was published in the *J. Amer. Math. Soc.*, **28** (1971), 453—455.

Introduce an arbitrary orthonormal system $\{g_k(y)\}$ defined on a measure space Y with positive measure ν, then

$$\lambda_{\nu_{n(x)}}^{-\frac{1}{2}} s_{\nu_{n(x)}}(x) = \int_Y \sum_{k=0}^{\nu_n} a_k g_k(t) \cdot \lambda_{\nu_{n(x)}}^{-\frac{1}{2}} \sum_{k=0}^{\nu_{n(x)}} g_k(t) f_k(x)\, d\nu(t).$$

So we obtain by Schwarz's inequality

$$\left| \int_E \lambda_{\nu_{n(x)}}^{-\frac{1}{2}} s_{\nu_{n(x)}}(x)\, d\mu(x) \right| \leqq$$

$$\leqq \left\{ \int_Y \left[\sum_{k=0}^{\nu_n} a_k g_k(t) \right]^2 d\nu(t) \cdot \int_Y \left[\int_E \lambda_{\nu(n)x}^{-\frac{1}{2}} \sum_{k=0}^{\nu_{n(x)}} g_k(t) f_k(x)\, d\mu(x) \right]^2 d\nu(t) \right\}^{\frac{1}{2}} \leqq$$

$$\leqq \left\{ \sum_{k=0}^{\infty} a_k^2 \right\}^{\frac{1}{2}} \left\{ \int_E \int_E \int_Y \lambda_{\nu_{n(x)}}^{-\frac{1}{2}} \sum_{k=0}^{\nu_{n(x)}} g_k(t) f_k(x) \cdot \lambda_{\nu_{n(y)}}^{-\frac{1}{2}} \sum_{k=0}^{\nu_{n(y)}} g_k(t) f_k(y)\, d\mu(x)\, d\mu(y)\, d\nu(t) \right\}^{\frac{1}{2}} =$$

$$= O(1) \left\{ \int_E \int_E \lambda_{\nu_n(x)}^{-\frac{1}{2}} \lambda_{\nu_n(y)}^{-\frac{1}{2}} \left| \sum_{k=0}^{\nu_{n(x,y)}} f_k(x) f_k(y) \right| d\mu(x)\, d\mu(y) \right\}^{\frac{1}{2}},$$

where $n(x, y) = \min\{n(x), n(y)\}$. As the sum in the last integrand equals $K_{\nu_{n(x,y)}}(x, y)$, it follows

$$\left| \int_E \lambda_{\nu_{n(x)}}^{-\frac{1}{2}} s_{\nu_{n(x)}}(x)\, d\mu(x) \right| =$$

$$= O(1) \left\{ \int_E \int_E \lambda_{\nu_{n(x)}}^{-1} |K_{\nu_{n(x)}}(x, y)|\, d\mu(x)\, d\mu(y) + \int_E \int_E \lambda_{\nu_{n(y)}}^{-1} |K_{\nu_{n(y)}}(x, y)|\, d\mu(x)\, d\mu(y) \right\}^{\frac{1}{2}} =$$

$$= O(1) \left\{ \int_E \lambda_{\nu_{n(x)}}^{-1} L_{\nu_{n(x)}}(x)\, d\mu(x) + \int_E \lambda_{\nu_{n(y)}}^{-1} L_{\nu_{n(y)}}(y)\, d\mu(y) \right\}^{\frac{1}{2}} = O(1).$$

Since the sequence $\{\lambda_{\nu_{n(x)}}^{-\frac{1}{2}} s_{\nu_{n(x)}}(x)\}$ is increasing, it follows by B. Levi's theorem that

$$\lambda_{\nu_{n(x)}}^{-\frac{1}{2}} s_{\nu_{n(x)}}(x) < \infty \quad \text{a.e.}$$

The same is true for the sequence $\{-\lambda_{\nu_{n(x)}}^{-\frac{1}{2}} s_{\nu_{n(x)}}(x)\}$; hence

$$\lambda_{\nu_{n(x)}}^{-\frac{1}{2}} |s_{\nu_{n(x)}}(x)| = O_x(1) \qquad \text{a.e,}$$

which contains our statement.

Theorem 2. *If the Lebesgue functions $L_n(x)$ are uniformly bounded on the measurable set E of finite measure and $\Sigma a_n^2 < \infty$, then the series $\Sigma a_n f_n(x)$ is convergent on E a. e.*

Indeed, $\Sigma a_n^2 < \infty$ implies $\Sigma a_n^2 \mu_n < \infty$ with an appropriate increasing sequence $\{\mu_n\}$ of positive numbers tending to infinity. Then we get by partial summation for every $m \geqq n$:

$$|s_m(x) - s_n(x)| = \left| \sum_{k=n+1}^{m} \mu_k^{-\frac{1}{2}} \mu_k^{\frac{1}{2}} a_k f_k(x) \right| \leqq$$

$$\leqq \sum_{k=n+1}^{m-1} (\mu_k^{-\frac{1}{2}} - \mu_{k+1}^{-\frac{1}{2}}) \left| \sum_{l=0}^{k} \mu_l^{\frac{1}{2}} a_l f_l(x) \right| + \mu_{n+1}^{-\frac{1}{2}} \left| \sum_{l=0}^{n} \mu_l^{\frac{1}{2}} a_l f_l(x) \right| + \mu_m^{-\frac{1}{2}} \left| \sum_{l=0}^{m} \mu_l^{\frac{1}{2}} a_l f_l(x) \right|.$$

Since $\Sigma a_n^2 \mu_n < \infty$ and $L_n(x) = O(1)$ for $x \in E$, we can apply Theorem 1 with $\nu_n = n$ and $\lambda_n = 1$ for every n. It follows then

$$\sum_{l=0}^{k} \mu_l^{\frac{1}{2}} a_l f_l(x) = O_x(1)$$

for every k and almost all $x \in E$; hence $s_m(x) - s_n(x) = o_x(1)$ a.e.

Theorem 3. *Suppose $L_{\nu_n}(x) = O(\lambda_{\nu_n})$ for every $x \in E$ and $\Sigma a_n^2 \lambda_n < \infty$. If also condition (1) is satisfied, then the sequence $\{s_{\nu_n}(x)\}$ of partial sums of the series $\Sigma a_n f_n(x)$ converges on E a.e.*

Set

$$S_n(x) = \sum_{k=0}^{n} \lambda_k^{\frac{1}{2}} \mu_k^{\frac{1}{2}} a_k f_k(x)$$

with an appropriate increasing sequence $\{\mu_n\}$ of positive numbers tending to infinity, and $\Sigma a_n^2 \lambda_n \mu_n < \infty$. We proceed as in the proof of Theorem 2:

$$|s_{\nu_m}(x) - s_{\nu_n}(x)| \leqq \sum_{k=\nu_n+1}^{\nu_m - 1} [(\lambda_k \mu_k)^{-\frac{1}{2}} - (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}}] |S_k(x)| + \tag{3}$$
$$+ (\lambda_{\nu_n+1} \mu_{\nu_n+1})^{-\frac{1}{2}} |S_{\nu_n}(x)| + (\lambda_{\nu_m} \mu_{\nu_m})^{-\frac{1}{2}} |S_{\nu_m}(x)|.$$

Because of $\Sigma a_k^2 \lambda_k \mu_k < \infty$ we have by Theorem 1

$$S_{\nu_n}(x) = O_x(\lambda_{\nu_n}^{\frac{1}{2}}) \quad \text{and} \quad S_{\nu_m}(x) = O_x(\mu_{\nu_m}^{\frac{1}{2}}) \quad \text{a.e.}$$

So the last two terms in (3) have the order of magnitude $o_x(1)$ a.e. on E. Regarding the first sum on the right hand side consider the series

$$S = \sum_{k=0}^{\infty} [(\lambda_k \mu_k)^{-\frac{1}{2}} - (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}}] \int_E |S_k(x)| \, d\mu(x).$$

Apply condition (1) with $b_k = (\lambda_k \mu_k)^{\frac{1}{2}}$, then the integrals on the right hand side are of order $O(1)$, hence

$$S = O(1) \sum_{k=0}^{\infty} [(\lambda_k \mu_k)^{-\frac{1}{2}} - (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}}] < \infty.$$

B. Levi's theorem implies the convergence a.e. of the series

$$\sum_{k=0}^{\infty} [(\lambda_k \mu_k)^{-\frac{1}{2}} - (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}}] S_k(x),$$

so we get from (3) $s_{\nu_m}(x) - s_{\nu_n}(x) = o_x(1)$ on E a. e.

Remark. Condition (1) can be weakened. We chose it only to get a simple and clear form of Theorem 3. But it could be replaced e.g. by

$$\int_E |S_k(x)|\, d\mu(x) = O\left(\lambda_k^{\frac{1}{2}-\varepsilon} \mu_k\right) \qquad (\varepsilon > 0)$$

supposing also that $\{1/\lambda_n\}$ is convex. It is easy to see that the series S would converge also under this condition.

As application of Theorem 3 we prove one of our results concerning multiplicatively orthogonal series, [2]. A system $\{\varphi_n(x)\}$ is called multiplicatively orthogonal on the measurable set E, if every finite product of different φ_k's has zero integral on E. That is, setting the product system $\psi_0(y) \equiv 1$ and $\psi_n(x) = \varphi_{m_1+1}(x)\varphi_{m_2+1}(x)\dots\varphi_{m_k+1}(x)$ for $n = 2^{m_1} + 2^{m_2} + \dots + 2^{m_k}$, we have

$$\int_E \psi_n(x)\, d\mu(x) = 0 \qquad (n \geqq 1).$$

Theorem 4. *Let $\{\varphi_n(x)\}$ be a multiplicatively orthogonal system defined on a measurable set E of finite measure. If $|\varphi_n(x)| \leqq M_n$, then $\Sigma c_n^2 M_n^2 < \infty$ implies the convergence a.e. on E of the series $\Sigma c_n \varphi_n(x)$.*

Denote by $\{f_n(x)\}$ the above defined product system of $\{\varphi_n(x)/M_n\}$ and set $a_n = c_{\nu+1} M_{\nu+1}$ for $n = 2^\nu$, and $a_n = 0$ for $n \neq 2^\nu$. Then we may write

$$s_{2^n-1}(x) = \sum_{k=0}^{2^n-1} a_k f_k(x) = \sum_{k=1}^{n} c_k M_k \cdot \frac{1}{M_k} \varphi_k(x).$$

We apply Theorem 3 with $\nu_n = 2^n - 1$ and $\lambda_n = 1$ ($n = 0, 1, \dots$). The Lebesgue functions $L_{2^n-1}(x)$ of the system $\{f_n(x)\}$ defined in this way are uniformly bounded on E, because

$$K_{2^n-1}(t, x) = \sum_{k=0}^{2^n-1} f_k(t) f_k(x) = \prod_{k=1}^{n} \left(1 + \frac{\varphi_k(t)\varphi_k(x)}{M_k^2}\right) \geqq 0,$$

and hence

$$L_{2^n-1}(x) = \int_E \sum_{k=0}^{2^n-1} f_k(t) f_k(x)\, d\mu(t) = \int_E d\mu(t).$$

Thus we have only to show that (1) is also satisfied. Choose, for this aim, $\{b_k\}$

arbitrarily so that $\Sigma a_k^2 b_k^2 < \infty$. Then, for every n of the form $2^m + p$ with $0 \leqq p < 2^m$ we have

$$\int_E \left| \sum_{k=0}^{n} a_k b_k f_k(x) \right| d\mu(x) = \left| \int_E \sum_{\nu=0}^{m} c_{\nu+1} b_{2^\nu} \cdot \frac{1}{M_{\nu+1}} \varphi_{\nu+1}(x)\, d\mu(x) \right| \leqq$$

$$\leqq \left\{ \int_E d\mu(x) \int_E \left[\sum_{\nu=0}^{m} c_{\nu+1} b_{2^\nu} \cdot \frac{1}{M_{\nu+1}} \varphi_{\nu+1}(x) \right]^2 d\mu(x) \right\}^{\frac{1}{2}} =$$

$$= O(1) \left\{ \sum_{\nu=0}^{m} c_{\nu+1}^2 b_{2^\nu}^2 \right\}^{\frac{1}{2}} = O(1) \left\{ \sum_{k=0}^{\infty} a_k^2 b_k^2 \right\}^{\frac{1}{2}} = O(1).$$

Hence condition (1) is satisfied and our statement proved.

3. The summation problem

Theorem 5. *If $\Sigma a_k^2 < \infty$ and the Lebesgue functions $L_n^1(x)$ satisfy the condition $L_n^1(x) = O(\lambda_n)$ uniformly on the measurable set E of finite measure, then the sums*

$$\sigma_n(x) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1} \right) a_k f_k(x)$$

have the order of magnitude $O_x(\lambda_n^{\frac{1}{2}})$ on E, a.e.

Denote by n_x the least index m $(\leqq n)$ for which

$$\lambda_m^{-\frac{1}{2}} \sigma_m(x) = \max_{0 \leqq k \leqq n} \lambda_k^{-\frac{1}{2}} \sigma_k(x),$$

and set $n_{x,y} = \min(n_x, n_y)$. Let be $\{g_k(y)\}$ an arbitrary orthonormal system defined in a measure space (Y, ν). Then

$$\left| \int_E \lambda_{n_x}^{-\frac{1}{2}} \sigma_{n_x}(x)\, d\mu(x) \right| = \left| \int_E \int_Y \sum_{k=0}^{n} a_k g_k(t) \cdot \lambda_{n_x}^{-\frac{1}{2}} \sum_{k=0}^{n_x} \left(1 - \frac{k}{n_x+1} \right) g_k(t) f_k(x)\, d\nu(t)\, d\mu(x) \right| \leqq$$

$$\leqq \left\{ \int_Y \left[\sum_{k=0}^{n} a_k g_k(t) \right]^2 d\nu(t) \int_E \int_E \int_Y \lambda_{n_x}^{-\frac{1}{2}} \lambda_{n_y}^{-\frac{1}{2}} \sum_{k=0}^{n_x} \left(1 - \frac{k}{n_x+1} \right) g_k(t) f_k(x) \times \right.$$

$$\left. \times \sum_{k=0}^{n_y} \left(1 - \frac{k}{n_y+1} \right) g_k(t) f_k(x)\, d\mu(x)\, d\mu(y)\, d\nu(t) \right\}^{\frac{1}{2}} =$$

$$= O(1) \left\{ \int_E \int_E \lambda_{n_x}^{-\frac{1}{2}} \lambda_{n_y}^{-\frac{1}{2}} \sum_{k=0}^{n_{x,y}} \left(1 - \frac{k}{n_x+1} \right) \left(1 - \frac{k}{n_y+1} \right) f_k(x) f_k(y)\, d\mu(x)\, d\mu(y) \right\}^{\frac{1}{2}}.$$

Apply twice partial summation to the sum in the last integrand to get

$$\sum_{k=0}^{n_{x,y}}\left(\frac{1}{n_x+1}+\frac{1}{n_y+1}-\frac{2k+1}{(n_x+1)(n_y+1)}\right)K_k(x,y)=$$

$$=\frac{2}{(n_x+1)(n_y+1)}\sum_{k=0}^{n_{x,y}-1}(k+1)K_k^1(x,y)+$$

$$+n_{x,y}K_{n_{x,y}}^1(x,y)\left(\frac{1}{n_x+1}+\frac{1}{n_y+1}-\frac{2n_{x,y}-1}{(n_x+1)(n_y+1)}\right);$$

hence it follows

$$\left|\int_E \lambda_{n_x}^{-\frac{1}{2}}\sigma_{n_x}(x)\,d\mu(x)\right|=$$

$$=O(1)\left\{\int_E\int_E \lambda_{n_x}^{-1}\left[(n_x+1)^{-2}\sum_{k=0}^{n_x}(k+1)|K_k^1(x,y)|+|K_{n_x}^1(x,y)|\right]d\mu(x)\,d\mu(y)\right\}^{\frac{1}{2}}=$$

$$=O(1)\left\{\int_E (n_x+1)^{-2}\sum_{k=0}^{n_x}(k+1)\lambda_{n_x}^{-1}L_k^1(x)\,d\mu(x)+\int_E \lambda_{n_x}^{-1}L_{n_x}(x)\,d\mu(x)\right\}^{\frac{1}{2}}=O(1).$$

Reasoning as before in the proof of Theorem 1 we obtain

$$|\sigma_{n_x}(x)|=O_x(\lambda_{n_n}^{\frac{1}{2}})\quad\text{a.e.}$$

Theorem 6. *If the Lebesgue functions $L_n^1(x)$ are uniformly bounded on the measurable set E of finite measure and $\Sigma a_n^2<\infty$, then the series $\Sigma a_n f_n(x)$ is $(C, 1)$-summable a.e.*

The convergence of Σa_n^2 implies the existence of a sequence $\{\mu_n\}$ of positive numbers, concave from below and tending to infinity such that $\Sigma a_n^2\mu_n<\infty$. Denote by $\Delta\mu_n^{-\frac{1}{2}}$ and $\Delta^2\mu_n^{-\frac{1}{2}}$ the first and the second differences of $\{\mu_n^{-\frac{1}{2}}\}$, respectively. Put $\sigma_n(\sqrt{\mu}, x)$ the n-th $(C, 1)$ mean of the series $\Sigma a_n\sqrt{\mu_n}f_n(x)$. By a known identity (see e.g. [1], p. 72) we have

$$\sigma_m(x)-\sigma_n(x)=\sum_{k=0}^{m-1}\left(1-\frac{k}{m+1}\right)\Delta^2\mu_k^{-\frac{1}{2}}\cdot(k+1)\sigma_k(\sqrt{\mu}, x)- \tag{4}$$

$$-\sum_{k=0}^{n-1}\left(1-\frac{k}{n+1}\right)\Delta^2\mu_k^{-\frac{1}{2}}\cdot(k+1)\sigma_k(\sqrt{\mu}, x)+\frac{2}{m+1}\sum_{k=0}^{m-1}\Delta\mu_{k+1}^{-\frac{1}{2}}(k+1)\sigma_k(\sqrt{\mu}, x)-$$

$$-\frac{2}{n+1}\sum_{k=0}^{n-1}\Delta\mu_{k+1}^{-\frac{1}{2}}\cdot(k+1)\sigma_k(\sqrt{\mu}, x)+\mu_m^{-\frac{1}{2}}\sigma_m(\sqrt{\mu}, x)-\mu_n^{-\frac{1}{2}}\sigma_n(\sqrt{\mu}, x).$$

From Theorem 5 we get $\sigma_k(\sqrt{\mu}, x)=O_x(1)$ a.e.; hence the last two terms in (4) have the order of magnitude $o_x(1)$ a.e. Since the sequence $\{\mu_n^{-\frac{1}{2}}\}$ is convex and tends to zero, it follows $\Delta\mu_n^{-\frac{1}{2}}=o(n^{-1})$. Thus the third and fourth terms in (4) are also $o_x(1)$ a.e. In what concerns the first two terms, take into account that by Theorem 5 the series

$$\sum_{k=0}^{\infty} \Delta^2 \mu_k^{-\frac{1}{2}} \cdot (k+1) |\sigma_k(\sqrt{\mu}, x)| = O_x(1) \sum_{k=0}^{\infty} (k+1) \Delta^2 \mu_k^{-\frac{1}{2}}$$

converges a. e. because of the convexity of $\{\mu_k^{-\frac{1}{2}}\}$. The first two terms in (4), being the difference of the m-th and n-th $(C, 1)$ means of an a. e. convergent series, tend to zero a.e., consequently

$$\sigma_m(x) - \sigma_n(x) = o_x(1) \quad \text{a.e.} \quad (m > n).$$

Theorem 7. *Let $\{\lambda_n\}$ be an increasing sequence of positive numbers concave from below. Suppose $L_n^1(x) = O(\lambda_n)$ for every $x \in E$ and $\Sigma a_n^2 \lambda_n < \infty$. If condition* (1) *is also satisfied, then the series $\Sigma a_n f_n(x)$ is $(C, 1)$-summable on E almost everywhere.*

Choose first a sequence $\{\mu_n\}$ of positive numbers concave from below and tending to infinity, such that $\Sigma a_n^2 \lambda_n \mu_n < \infty$ and that $\{\lambda_n \mu_n\}$ be concave from below. Using the same notations as in the proof of Theorem 6, with $\lambda_n \mu_n$ instead of μ_n, we get first

$$\sigma_n(x) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n+1}\right) \Delta^2 (\lambda_k \mu_k)^{-\frac{1}{2}} (k+1) \sigma_k(\sqrt{\lambda \mu}, x) + \tag{5}$$

$$+ \frac{2}{n+1} \sum_{k=0}^{n-1} \Delta (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}} (k+1) \sigma_k(\sqrt{\lambda \mu}, x) + (\lambda_n \mu_n)^{-\frac{1}{2}} \sigma_n(\sqrt{\lambda \mu}, x).$$

By Theorem 5 we have $\sigma_n(\sqrt{\lambda \mu}, x) = O_x(\lambda_n^{-\frac{1}{2}})$ a.e.; hence the last term on the right hand side is $o_x(1)$ a. e. As to the second, it follows by condition (1):

$$\sum_{k=0}^{\infty} \Delta (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}} \int_E |\sigma_k(\sqrt{\lambda \mu}, x)| \, d\mu(x) < \infty,$$

thus $\Sigma \Delta (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}} |\sigma_k(\sqrt{\lambda \mu}, x)|$ converges a. e. This implies the existence of an index $N = N(x, \varepsilon)$ such that for an arbitrary $\varepsilon > 0$

$$\sum_{k=N}^{\infty} \Delta (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}} |\sigma_k(\sqrt{\lambda \mu}, x)| < \frac{\varepsilon}{4} \quad \text{a.e.}$$

Therefore we get for sufficiently large n and almost all $x \in E$

$$\left| \frac{2}{n+1} \sum_{k=0}^{n-1} \Delta (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}} (k+1) \sigma_k(\sqrt{\lambda \mu}, x) \right| \leqq$$

$$\leqq \frac{2}{n+1} \sum_{k=0}^{N-1} \Delta (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}} (k+1) |\sigma_k(\sqrt{\lambda \mu}, x)| +$$

$$+ 2 \sum_{k=N}^{n-1} \Delta (\lambda_{k+1} \mu_{k+1})^{-\frac{1}{2}} |\sigma_k(\sqrt{\lambda \mu}, x)| = O_x(1) \frac{N^2}{n+1} + \frac{\varepsilon}{2} < \varepsilon.$$

In other words, the second term on the right hand side of (5) is $o_x(1)$ on E a.e. Concerning the first term we get

$$\sum_{k=0}^{\infty} \Delta^2(\lambda_k\mu_k)^{-\frac{1}{2}}(k+1)\int_E \left|\sigma_k(\sqrt{\lambda\mu}, x)\right| d\mu(x) < \infty$$

by condtion (1) and the convexity of $\{(\lambda_k\mu_k)^{-\frac{1}{2}}\}$, i. e. the series

$$\Sigma\Delta^2(\lambda_k\mu_k)^{-\frac{1}{2}}(k+1)\sigma_k(\sqrt{\lambda\mu}, x)$$

converges a. e. The first term, being about the $(n-1)$th $(C, 1)$ mean of this series, is also convergent for almost all $x \in E$. Thus we see by (5) that $\{\sigma_n(x)\}$ is decomposed for almost all $x \in E$ in a convergent sequence and two terms of order $o_x(1)$. Consequently, $\{\sigma_n(x)\}$ converges a.e. as we have stated.

Remarks. 1. One can easily see that Theorems 2 and 6 cannot be improved. Indeed, *if $a_0, a_1, \ldots$ are arbitrary real numbers such that $\Sigma a_n^2 = \infty$, there exists a system $\{f_n(x)\}$ of continuous functions in* (0, 1) *with uniformly bounded Lebesgue functions such that $\Sigma a_n f_n(x)$ is nowhere summable by any regular positive Toeplitz method.*

Choose $E = [0, 1]$ with the ordinary Lebesgue measure μ. The system

$$f_n(x) = a_n\left(\sum_{\nu=0}^{n} a_\nu^2\right)^{-1} \qquad (0 \leqq x \leqq 1)$$

has the required property. Indeed, for every $x \in [0, 1]$ we have

$$L_n(x) = \int_0^1 \sum_{k=0}^{n} a_k^2\left(\sum_{\nu=0}^{k} a_\nu^2\right)^{-2} dt \leqq \sum_{k=0}^{\infty} a_k^2\left(\sum_{\nu=0}^{k} a_\nu^2\right)^{-2}.$$

The last series is convergent, and hence $L_n(x) = O(1)$ uniformly in [0, 1]. But

$$s_n(x) = \sum_{k=0}^{n} a_k^2\left(\sum_{\nu=0}^{k} a_\nu^2\right)^{-1} \to \infty.$$

The terms of $\Sigma a_n f_n(x)$ are positive, thus the series is not summable by any regular positive Toeplitz method.

2. Theorems 5, 6, and 7 can be generalized in that way that their statements remain valid for any (C, α)-summation $(\alpha > 0)$, if we substitute $L_n^1(x)$ by the corresponding Lebesgue functions $L_n^\alpha(x)$. The proofs are similar, but longer, because of the more intricate computations with (C, α) means. The technique could be copied from [6].

3. One can see that condition (1) could be weakened also in Theorem 7 exactly as we indicated in our remark to Theorem 3. Moreover we could substitute the condition of concavity of the sequence $\{\lambda_n\}$ by other, somewhat less pretentious conditions. But it seemed us that a simple form of Theorem 7 shows clearer the essence than any other more sophisticated statement.

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On some inequalities concerning series of positive terms

By LÁSZLÓ LEINDLER in Szeged

Recently A. PRÉKOPA [1] has proved the integral inequality

$$(1)\qquad \int_{-\infty}^{\infty} \sup_{x+y=t} f(x)g(y)\,dt \geqq 2\Big(\int_{-\infty}^{\infty} f^2(x)\,dx\Big)^{1/2}\Big(\int_{-\infty}^{\infty} g^2(y)\,dy\Big)^{1/2},$$

where $f(x)$ and $g(y)$ are arbitrary non-negative measurable functions.

It seems worth while to observe that the formal analogue

$$\sum_{n=-\infty}^{\infty} \sup_{k+l=n} a_k b_l \geqq 2\left(\sum_{k=-\infty}^{\infty} a_n^2\right)^{1/2}\left(\sum_{l=-\infty}^{\infty} b_l^2\right)^{1/2}$$

of the inequality (1) is in general not true. (See e. g. the sequences $a_0=b_0=1$ and $a_n=b_n=0$ if $n\neq 0$.) But we will show that without the factor 2 the inequality does hold, i. e. we have

$$(2)\qquad \sum_{n=-\infty}^{\infty} \sup_{k+l=n} a_k b_l \geqq \left(\sum_{k=-\infty}^{\infty} a_k^2\right)^{1/2}\left(\sum_{l=-\infty}^{\infty} b_l^2\right)^{1/2}$$

for any non-negative sequences $\{a_n\}$ and $\{b_n\}$.

First we give a very short and simple proof of (2). Next we generalize (2) as follows:

Theorem. *Suppose that* $1\leqq r,\ s\leqq\infty$ *and* $\frac{1}{r}+\frac{1}{s}=1+\frac{1}{\gamma}$, *where* $1\leqq\gamma\leqq\infty$. *Then for non-negative* a_n, b_n *we have*

$$(3)\qquad \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} a_k^{\gamma} b_{n-k}^{\gamma}\right)^{1/\gamma} \geqq \left(\sum_{k=-\infty}^{\infty} a_k^r\right)^{1/r}\left(\sum_{n=-\infty}^{\infty} b_n^s\right)^{1/s}. \text{ *)}$$

---

*) If $c_k\geqq 0$ and $\gamma=\infty$, then $\left\{\sum_{k=-\infty}^{\infty} c_k^{\gamma}\right\}^{1/\gamma}$ means $\sup_k c_k$.

Let us formulate the special case $\gamma=\infty$ of (3) as a

Corollary. *If* $p, q\geqq 1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\sum_{n=-\infty}^{\infty} \sup_k a_k b_{n-k} \geqq \left(\sum_{n=-\infty}^{\infty} a_n^p\right)^{1/p} \left(\sum_{n=-\infty}^{\infty} b_n^q\right)^{1/q}. \tag{4}$$

Our theorem can be generalized from two to any finite number of series with a straightforward generalization of the proof which follows.

E.g. *we have*

$$\sum_{n=-\infty}^{\infty}\left[\sum_{i+j+k=n}(a_i b_j c_k)^\gamma\right]^{1/\gamma} \geqq \left[\sum_{i=-\infty}^{\infty} a_i^r\right]^{1/r}\left[\sum_{j=-\infty}^{\infty} b_j^s\right]^{1/s}\left[\sum_{k=-\infty}^{\infty} c_k^t\right]^{1/t},$$

where a_n, b_n, c_n *are non-negative and* $1\leqq r,\ s,\ t\leqq\infty$; $\frac{1}{r}+\frac{1}{s}+\frac{1}{t}=2+\frac{1}{\gamma}$ $(1\leqq\gamma\leqq\infty)$.

The integral analogue of our result will be published in another paper. The method of proof to be given there is quite different from that of Prékopa and also slightly different from that one given for series in the present paper.

Proof of (2). Denote

$$\|a_n\| = \left(\sum_{n=-\infty}^{\infty} a_n^2\right)^{1/2}, \quad \|b_n\| = \left(\sum_{n=-\infty}^{\infty} b_n^2\right)^{1/2} \quad \text{and} \quad c_n = \sup_k a_k b_{n-k}.$$

We may assume that $0<\Sigma c_n<\infty$. This assumption implies that not all a_n and b_n vanish, and $0<\|a_n\|<\infty$ and $0<\|b_n\|<\infty$, indeed, for any k and n, $c_n\geqq a_k b_{n-k}$. Taking the Cauchy product of the series Σa_n^2 and Σb_n^2 we obtain

$$\|a_n\|^2\|b_n\|^2 = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_k^2 b_{n-k}^2 \leqq \sum_{n=-\infty}^{\infty} c_n \sum_{k=-\infty}^{\infty} a_k b_{n-k} \leqq \sum_{n=-\infty}^{\infty} c_n \|a_n\|\cdot\|b_n\|.$$

Hence (2) follows evidently.

Proof of (3). We may also assume that the sum on the left-hand side of (3) has finite value and that not all a_n and b_n vanish. If $a_\mu>0$, then we have

$$\left[\sum_{k=-\infty}^{\infty}(a_k b_{n-k})^\gamma\right]^{1/\gamma} \geqq a_\mu b_{n-\mu} \quad \text{for any} \quad n,$$

and hence

$$\sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty}(a_k b_{n-k})^\gamma\right]^{1/\gamma} \geqq a_\mu \sum_{n=-\infty}^{\infty} b_n.$$

Therefore $\Sigma b_n < \infty$. For analogous reasons, $\Sigma a_n < \infty$. Since $r \geqq 1$ and $s \geqq 1$ we also have

(5) $$A \equiv \left[\sum_{n=-\infty}^{\infty} a_n^r\right]^{1/r} < \infty \quad \text{and} \quad B \equiv \left[\sum_{n=-\infty}^{\infty} b_n^s\right]^{1/s} < \infty.$$

If $r=\infty$ or $s=\infty$, then $\gamma=\infty$ and $s=1$ or $r=1$, respectively. In these cases (3) holds. If e.g. $r=\infty$, then by (5) there exists ν such that $a_\nu = \sup a_k$; thus the inequality

$$a_\nu b_n \leqq \sup_k a_k b_{n+\nu-k}$$

holds for all n; hence we obtain that

$$\sum_{n=-\infty}^{\infty} a_\nu b_n \leqq \sum_{n=-\infty}^{\infty} \sup_k a_k b_{n+\nu-k}$$

and this is what was to be proved.

If both r and s are finite we set

(6) $$c_n = a_n/A \quad \text{and} \quad d_n = b_n/B,$$

and we have

(7) $$\sum_{n=-\infty}^{\infty} c_n^r = 1 \quad \text{and} \quad \sum_{n=-\infty}^{\infty} d_n^s = 1.$$

Taking the Cauchy product of these two series we obtain

(8) $$\sum_{n=-\infty}^{\infty} I_n = 1, \quad \text{where} \quad I_n = \sum_{k=-\infty}^{\infty} c_k^r d_{n-k}^s.$$

Next we prove that

(9) $$I_n \leqq \left[\sum_{k=-\infty}^{\infty} (c_k d_{n-k})^\gamma\right]^{1/\gamma}.$$

If $\gamma=1$, then $r=s=1$, and thus in (9) equality holds. If $1<\gamma\leqq\infty$, set $\gamma' = \dfrac{\gamma}{\gamma-1}$ $(1\leqq\gamma'<\infty)$. Then by the inequality of Hölder,

$$I_n = \sum_{k=-\infty}^{\infty} c_k d_{n-k} c_k^{r-1} d_{n-k}^{s-1} \leqq J_n \left[\sum_{k=-\infty}^{\infty} (c_k d_{n-k})^\gamma\right]^{1/\gamma},$$

where

$$J_n = \left[\sum_{k=-\infty}^{\infty} c_k^{(r-1)\gamma'} d_{n-k}^{(s-1)\gamma'}\right]^{1/\gamma'},$$

thus we have to prove that $J_n \leqq 1$. If $r=1$ then $(s-1)\gamma' = s$, and if $s=1$ then

$(r-1)\gamma' = r$; thus in these cases $J_n \leqq 1$ follows immediately from (7). If both r and s are greater than 1, we can use (7) through the inequality

$$xy \leqq \frac{x^p}{p} + \frac{y^q}{q} \qquad (x, y \geqq 0),$$

applied to $x = c_k^{(r-1)\gamma'}$, $y = d_{n-k}^{(s-1)\gamma'}$, $p = \dfrac{r}{(r-1)\gamma'}$ and $q = \dfrac{s}{(s-1)\gamma'}$ (note that $\dfrac{1}{p} + \dfrac{1}{q} = 1$); then we get

$$J_n^{\gamma'} \leqq \sum_{k=-\infty}^{\infty} \left(\frac{1}{p} c_k^r + \frac{1}{q} d_{n-k}^s \right) = 1.$$

Now by (5), (7), (8) and (9) we have

$$1 = \sum_{n=-\infty}^{\infty} I_n \leqq \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} \left(\frac{a_k}{A} \right)^{\gamma} \left(\frac{b_{n-k}}{B} \right)^{\gamma} \right]^{1/\gamma} = \frac{1}{A \cdot B} \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} (a_k b_{n-k})^{\gamma} \right]^{1/\gamma},$$

which implies (3).

The proof is now complete.

My grateful acknowledgement is due to Professor Béla Szőkefalvi-Nagy for stimulating conversations.

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On a linear transformation in the theory of probability

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1. Introduction. In the theory of random fluctuations we frequently encounter the following problem: A sequence of mutually independent and identically distributed real random variables $\{\xi_n;\ n=1, 2, \ldots\}$ is given. We define a sequence of random variables $\{\eta_n;\ n=0, 1, 2, \ldots\}$ by the recurrence formula $\eta_n = \max(0, \eta_{n-1}+\xi_n)$ $(n=1, 2, \ldots)$, where η_0 is a nonnegative random variable which is independent of the sequence $\{\xi_n\}$. The problem is to find the distribution function or the Laplace—Stieltjes transform of η_n for every $n=1, 2, \ldots$. We have several methods at our disposal for finding the generating function

$$\sum_{n=0}^{\infty} \mathrm{E}\{e^{-s\eta_n}\}\varrho^n$$

for $\mathrm{Re}(s)\geqq 0$ and $|\varrho|<1$; namely, analytical methods (F. POLLACZEK [12], [13], I. J. GOOD [6], J. H. B. KEMPERMAN [7]), algebraic methods (G. BAXTER [2], [3], J. G. WENDEL [18], [19], J. F. C. KINGMAN [8], [9], G.-C. ROTA [14]), combinatorial methods (E. S. ANDERSEN [1], F. SPITZER [16], W. FELLER [5], L. TAKÁCS [17]), and the method of factorization (see e.g. J. H. B. KEMPERMAN [7] and A. A. BOROVKOV [4]). The method of factorization has been introduced by N. WIENER and E. HOPF [21] for solving integral equations. (See also F. SMITHIES [15], H. WIDOM [20], and N. I. MUSKHELISHVILI [10].) It seems that all the existing methods have certain limitations. The analytic method of Pollaczek is constructive and gives the solution in a closed form; however, certain restrictions should be imposed on the distribution function of ξ_n. Furthermore, since the solution appears as a solution of a singular integral equation, the uniqueness of the solution should be proved. The algebraic methods are mostly descriptive, and even in the particular case when $\mathbf{P}\{\eta_0=0\}=1$, the solution does not appear in a closed form. In general, combinatorial methods do not provide the solution in a closed form either, but fortunately, in some partic-

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ular cases, we can obtain explicit expressions for $\mathbf{P}\{\eta_n \leqq x\}$ $(n=1, 2, \ldots)$. The method of factorization is mostly restricted to the case of $\mathbf{P}\{\eta_0=0\}=1$.

In what follows we shall consider a more general problem than the one mentioned above, namely, the problem of finding a sequence of functions $\Gamma_n(s)$ $(n=1, 2, \ldots)$ defined for $\operatorname{Re}(s)=0$ by a recurrence relation $\Gamma_n(s)=\mathbf{T}\{\gamma(s)\Gamma_{n-1}(s)\}$, where $\gamma(s)$ and $\Gamma_0(s)$ are elements of a commutative Banach algebra $\mathbf{R}$, $\mathbf{T}$ is a projection and $\mathbf{T}\{\Gamma_0(s)\}=\Gamma_0(s)$. We shall define $\mathbf{R}$ in such a way that on the one hand $\mathbf{R}$ is large enough to contain all the important functions arising in fluctuation theory and on the other hand $\mathbf{R}$ is small enough to allow an explicit representation of the transformation $\mathbf{T}$, which is suitable for calculations. We shall provide a constructive method for finding the generating function of $\Gamma_n(s)$ $(n=0, 1, 2, \ldots)$, and we shall obtain the solution in a closed form. As a byproduct we obtain the method of factorization and we shall show how it can be applied in the general case.

2. A Banach algebra R. Denote by $\mathbf{R}$ the space of functions $\Phi(s)$ defined for $\operatorname{Re}(s)=0$ on the complex plane, which can be represented in the form

$$\Phi(s)=\mathbf{E}\{\zeta e^{-s\eta}\}, \tag{1}$$

where ζ is a complex (or real) random variable with $\mathbf{E}\{|\zeta|\}<\infty$, and η is a real random variable. The function $\Phi(s)$ is uniquely determined by the joint distribution of ζ and η. However, there are infinitely many possible distributions which yield the same $\Phi(s)$. It follows from (1) that $|\Phi(s)|\leqq\mathbf{E}\{|\zeta|\}$ for $\operatorname{Re}(s)=0$.

Let us define the norm of $\Phi(s)$ by

$$\|\Phi\|=\inf_{\zeta}\mathbf{E}\{|\zeta|\} \tag{2}$$

where the infimum is taken for all ζ for which (1) holds (with a suitable η). Obviously, $|\Phi(s)|\leqq\|\Phi\|$ for $\operatorname{Re}(s)=0$.

We have $\|\Phi\|\geqq 0$, and $\|\Phi\|=0$ if and only if $\Phi(s)\equiv 0$. If α is a complex (or real) number and $\Phi(s)\in\mathbf{R}$, then $\alpha\Phi(s)\in\mathbf{R}$ and $\|\alpha\Phi\|=|\alpha|\,\|\Phi\|$. Furthermore, if $\Phi_1(s)\in\mathbf{R}$ and $\Phi_2(s)\in\mathbf{R}$, then $\Phi_1(s)+\Phi_2(s)\in\mathbf{R}$ and $\|\Phi_1+\Phi_2\|\leqq\|\Phi_1\|+\|\Phi_2\|$. The last statement can be proved as follows:

For any $\varepsilon>0$ let $\Phi_1(s)=\mathbf{E}\{\zeta_1 e^{-s\eta_1}\}$, where $\mathbf{E}\{|\zeta_1|\}\leqq\|\Phi_1\|+\varepsilon$, and let $\Phi_2(s)=\mathbf{E}\{\zeta_2 e^{-s\eta_2}\}$, where $\mathbf{E}\{|\zeta_2|\}\leqq\|\Phi_2\|+\varepsilon$. Let ν be a random variable which is independent of (ζ_1,η_1) and (ζ_2,η_2), and for which $\mathbf{P}\{\nu=1\}=\mathbf{P}\{\nu=2\}=\frac{1}{2}$. Let us define $\zeta=2\zeta_\nu$ and $\eta=\eta_\nu$. Then

$$\mathbf{E}\{\zeta e^{-s\eta}\}=\Phi_1(s)+\Phi_2(s) \quad\text{and}\quad \mathbf{E}\{|\zeta|\}=\mathbf{E}\{|\zeta_1|\}+\mathbf{E}\{|\zeta_2|\}<\infty. \tag{3}$$

Thus $\Phi_1(s)+\Phi_2(s)\in\mathbf{R}$, and $\|\Phi_1+\Phi_2\|\leqq\|\Phi_1\|+\|\Phi_2\|+2\varepsilon$. Since $\varepsilon>0$ is arbitrary, this proves the statement.

In what follows we shall not make use of the completeness of **R**. However, we can prove that **R** is complete, and therefore **R** is a Banach space.

Next we observe that if $\Phi_1(s)\in\mathbf{R}$ and $\Phi_2(s)\in\mathbf{R}$, then $\Phi_1(s)\Phi_2(s)\in\mathbf{R}$ and $\|\Phi_1\Phi_2\|\leqq\|\Phi_1\|\,\|\Phi_2\|$. To prove this let us define $\Phi_1(s)$ and $\Phi_2(s)$ in exactly the same way as above. However, let us assume now that (ζ_1, η_1) and (ζ_2, η_2) are independent and take $\zeta=\zeta_1\zeta_2$ and $\eta=\eta_1+\eta_2$. Then

$$\mathbf{E}\{\zeta e^{-s\eta}\} = \Phi_1(s)\Phi_2(s) \quad\text{and}\quad \mathbf{E}\{|\zeta|\} = \mathbf{E}\{|\zeta_1|\}\mathbf{E}\{|\zeta_2|\} < \infty. \tag{4}$$

Thus $\Phi_1(s)\Phi_2(s)\in\mathbf{R}$ and $\|\Phi_1\Phi_2\| \leqq (\|\Phi_1\|+\varepsilon)(\|\Phi_2\|+\varepsilon)$. Since $\varepsilon>0$ is arbitrary this proves the statement.

Accordingly, **R** is a commutative Banach algebra.

3. A linear tansformation T. Let us define a transformation **T** in **R** by

$$\mathbf{T}\{\Phi(s)\} = \Phi^+(s) = \mathbf{E}\{\zeta e^{-s\eta^+}\}, \tag{5}$$

where $\eta^+=\max(0, \eta)$. As we shall show explicity in Theorem 2, the function $\Phi^+(s)$ is independent of the particular representation (1) of $\Phi(s)$. Observe that $\Phi^+(s)$ is a regular function of s in the domain $\operatorname{Re}(s)>0$, and continuous for $\operatorname{Re}(s)\geqq 0$. Furthermore, $|\Phi^+(s)|\leqq\|\Phi\|$ for $\operatorname{Re}(s)\geqq 0$.

If α is a complex (or real) number and $\Phi(s)\in\mathbf{R}$, then $\mathbf{T}\{\alpha\Phi(s)\}=\alpha\mathbf{T}\{\Phi(s)\}$. If $\Phi_1(s)\in\mathbf{R}$ and $\Phi_2(s)\in\mathbf{R}$, then $\mathbf{T}\{\Phi_1(s)+\Phi_2(s)\} = \mathbf{T}\{\Phi_1(s)\}+\mathbf{T}\{\Phi_2(s)\}$. This follows immediately from the representation (3). Obviously, $\|\mathbf{T}\|=1$. Accordingly, **T** is a bounded linear transformation. Moreover, $\mathbf{T}^2=\mathbf{T}$, that is, **T** is a projection.

We note that if $\Phi_1(s)\in\mathbf{R}$ and $\Phi_2(s)\in\mathbf{R}$, and $\mathbf{T}\{\Phi_1(s)\}=\Phi_1(s)$ and $\mathbf{T}\{\Phi_2(s)\}=$ $=\Phi_2(s)$, then $\mathbf{T}\{\Phi_1(s)\Phi_2(s)\}=\Phi_1(s)\Phi_2(s)$. Furthermore, if $\Phi_1(s)\in\mathbf{R}$ and $\Phi_2(s)\in\mathbf{R}$, and $\mathbf{T}\{\Phi_1(s)\}=c_1$ and $\mathbf{T}\{\Phi_2(s)\}=c_2$, where c_1 and c_2 are complex (or real) constans, then $\mathbf{T}\{\Phi_1(s)\Phi_2(s)\}=c_1c_2$. These statements follow immediately from the representation (4).

4. A recurrence relation. The problem mentioned in the Introduction and many other problems in the theory of probability and stochastic processes can be reduced to the problem of finding a sequence of functions $\{\Gamma_n(s)\}$ satisfying a recurrence relation of the form

$$\Gamma_n(s)=\mathbf{T}\{\gamma(s)\Gamma_{n-1}(s)\}\ \ (n=1, 2, \ldots),\quad\text{with}\quad \mathbf{T}\{\Gamma_0(s)\}=\Gamma_0(s)\quad\text{and}\quad \gamma(s)\in\mathbf{R}.$$

To solve this problem we need the following auxiliary theorem.

Lemma. *Let $\Phi_n(s)\in\mathbf{R}$ for $n=0, 1, 2, \ldots$ and let a_n $(n=0, 1, 2, \ldots)$ be complex (or real) numbers. If*

$$\sum_{n=0}^{\infty}|a_n|\,\|\Phi_n\|<\infty,$$

then

$$\Psi(s) = \sum_{n=0}^{\infty} a_n \Phi_n(s) \in \mathbf{R} \quad \textit{and} \quad \mathbf{T}\{\Psi(s)\} = \sum_{n=0}^{\infty} a_n \mathbf{T}\{\Phi_n(s)\}. \tag{6}$$

Proof. If we refer to the facts that **R** is complete and **T** is continuous, then the Lemma follows immediately. However, we are not making use of the completeness of **R** and therefore a separate proof is required.

For $n=0, 1, 2, \ldots$ let $\Phi_n(s) = \mathbf{E}\{\zeta_n e^{-s\eta_n}\}$, where $\mathbf{E}\{|\zeta_n|\} \leqq 2\|\Phi_n\|$. Let ν be a discrete random variable which is independent of the sequence (ζ_n, η_n) $(n=0,1,2,\ldots)$ and which takes on nonnegative integral values with probabilities $\mathbf{P}\{\nu=n\}=p_n>0$ for $n=0, 1, 2, \ldots$. Define $\zeta = a_\nu \zeta_\nu / p_\nu$ and $\eta = \eta_\nu$. Then

$$\mathbf{E}\{\zeta e^{-s\eta}\} = \sum_{n=0}^{\infty} \mathbf{P}\{\nu = n\} \frac{a_n}{p_n} \mathbf{E}\{\zeta_n e^{-s\eta_n}\} = \sum_{n=0}^{\infty} a_n \Phi_n(s)$$

and

$$\mathbf{E}\{|\zeta|\} = \sum_{n=0}^{\infty} \mathbf{P}\{\nu = n\} \frac{|a_n|}{p_n} \mathbf{E}\{|\zeta_n|\} \leqq 2 \sum_{n=0}^{\infty} |a_n| \|\Phi_n\| < \infty.$$

Accordingly, $\Psi(s) = \mathbf{E}\{\zeta e^{-s\eta}\}$ and $\Psi(s) \in \mathbf{R}$. Furthermore, we have

$$\mathbf{T}\{\Psi(s)\} = \mathbf{E}\{\zeta e^{-s\eta^+}\} = \sum_{n=0}^{\infty} \mathbf{P}\{\nu = n\} \frac{a_n}{p_n} \mathbf{E}\{\zeta_n e^{-s\eta_n^+}\} = \sum_{n=0}^{\infty} a_n \mathbf{T}\{\Phi_n(s)\}$$

which is in agreement with (6). This completes the proof of the Lemma.

In particular, it follows from the Lemma that if $\Phi(s) \in \mathbf{R}$, then $e^{\varrho\Phi(s)} \in \mathbf{R}$ for any ϱ, and $[1-\varrho\Phi(s)]^{-1} \in \mathbf{R}$ and $\log[1-\varrho\Phi(s)] \in \mathbf{R}$ whenever $|\varrho| \|\Phi\| < 1$. If we form the power series expansions of these functions, then we can apply **T** term by term.

Theorem 1. *Let us suppose that* $\gamma(s) \in \mathbf{R}$, $\Gamma_0(s) \in \mathbf{R}$ *and* $\mathbf{T}\{\Gamma_0(s)\} = \Gamma_0(s)$. *Define* $\Gamma_n(s)$ *for* $n=1, 2, \ldots$ *by the recurrence relation*

$$\Gamma_n(s) = \mathbf{T}\{\gamma(s)\Gamma_{n-1}(s)\}. \tag{7}$$

If $|\varrho| \|\gamma\| < 1$, *then*

$$\sum_{n=0}^{\infty} \Gamma_n(s)\varrho^n = e^{-\mathbf{T}\{\log[1-\varrho\gamma(s)]\}} \mathbf{T}\{\Gamma_0(s) e^{-\log[1-\varrho\gamma(s)] + \mathbf{T}\{\log[1-\varrho\gamma(s)]\}}\} \tag{8}$$

for $\operatorname{Re}(s) \geqq 0$.

Proof. Let us denote the right-hand side of (8) by $U(s, \varrho)$. Obviously, $U(s, \varrho) \in \mathbf{R}$ and $\mathbf{T}\{U(s, \varrho)\} = U(s, \varrho)$. Now we shall show that $U(s, \varrho)$ satisfies the following equation

$$U(s, \varrho) - \varrho \mathbf{T}\{\gamma(s) U(s, \varrho)\} = \Gamma_0(s). \tag{9}$$

Let us introduce the function

$$h(s) = e^{\log[1-\varrho\gamma(s)] - \mathbf{T}\{\log[1-\varrho\gamma(s)]\}}$$

for Re $(s)=0$. It is obvious that $h(s)\in\mathbf{R}$, $1/h(s)\in\mathbf{R}$, and $\Gamma_0(s)/h(s)\in\mathbf{R}$. We can also see immediately that

(10) $$\mathbf{T}\{h(s)\}=1$$

and

(11) $$\mathbf{T}\left\{\frac{\Gamma_0(s)}{h(s)} - \mathbf{T}\frac{\Gamma_0(s)}{h(s)}\right\} = 0.$$

Now (10) and (11) imply that

(12) $$\mathbf{T}\left\{h(s)\left[\frac{\Gamma_0(s)}{h(s)} - \mathbf{T}\frac{\Gamma_0(s)}{h(s)}\right]\right\} = 0,$$

that is,

(13) $$\mathbf{T}\{[1-\varrho\gamma(s)]U(s,\varrho)\} = \Gamma_0(s)$$

whence (9) follows.

Let us expand $U(s, \varrho)$ in a power series as follows

(14) $$U(s,\varrho) = \sum_{n=0}^{\infty} U_n(s)\varrho^n.$$

This series is convergent if $|\varrho|\,\|\gamma\|<1$ and evidently $U_n(s)\in\mathbf{R}$ for $n=0, 1, 2, \dots$. If we put (14) into (9), then we obtain that $U_0(s)=\Gamma_0(s)$ and

(15) $$U_n(s) = \mathbf{T}\{\gamma(s)U_{n-1}(s)\}$$

for $n=1, 2, \dots$. Accordingly, the sequence $\{U_n(s)\}$ satisfies the same recurrence relation and the same initial condition as the sequence $\{\Gamma_n(s)\}$. Thus $U_n(s)=\Gamma_n(s)$ for $n=0, 1, 2, \dots$ which was to be proved.

We note that by the Lemma we have

$$\mathbf{T}\{\log[1-\varrho\gamma(s)]\} = -\sum_{n=1}^{\infty}\frac{\varrho^n}{n}\mathbf{T}\{[\gamma(s)]^n\}$$

for $|\varrho|\,\|\gamma\|<1$.

If, in particular, $\Gamma_0(s)\equiv 1$, then (8) reduces to

(16) $$\sum_{n=0}^{\infty}\Gamma_n(s)\varrho^n = e^{-\mathbf{T}\{\log[1-\varrho\gamma(s)]\}} = \exp\left\{\sum_{n=1}^{\infty}\frac{\varrho^n}{n}\mathbf{T}\{[\gamma(s)]^n\}\right\}$$

where $|\varrho|\,\|\gamma\|<1$.

The usefulness of formulas (8) and (16) depends on the applicability of the transformation $\mathbf{T}$. Our next aim is to give a method for finding $\mathbf{T}\{\Phi(s)\}$ for $\Phi(s)\in\mathbf{R}$ and, in particular, for finding $\mathbf{T}\{\log[1-\varrho\gamma(s)]\}$ for $\gamma(s)\in\mathbf{R}$ and $|\varrho|\,\|\gamma\|<1$.

5. A representation of T. If we know $\Phi(s)\in\mathbf{R}$ for $\operatorname{Re}(s)=0$, then $\Phi^+(s)=\mathbf{T}\{\Phi(s)\}$ is uniquely determined for $\operatorname{Re}(s)\geqq 0$ as a function which is regular in the domain $\operatorname{Re}(s)>0$ and continuous for $\operatorname{Re}(s)\geqq 0$. We can obtain $\Phi^+(s)$ explicitly by the following theorem.

Theorem 2. *If* $\Phi(s)\in\mathbf{R}$, *then for* $\operatorname{Re}(s)>0$ *we have*

$$\Phi^+(s) = \frac{1}{2}\Phi(0)+\lim_{\varepsilon\to 0}\frac{s}{2\pi i}\int_{L_\varepsilon}\frac{\Phi(z)}{z(s-z)}\,dz, \tag{17}$$

where the path of integration $L_\varepsilon(\varepsilon>0)$ *consists of the imaginary axis from* $z=-i\infty$ *to* $z=-i\varepsilon$ *and again from* $z=i\varepsilon$ *to* $z=i\infty$.

Proof. Let $C_\varepsilon^+(\varepsilon>0)$ be the path which consists of the imaginary axis from $z=-i\infty$ to $z=-i\varepsilon$, the semicircle $c_\varepsilon^+=\left\{z: z=\varepsilon e^{i\alpha},\ -\frac{\pi}{2}\leqq\alpha\leqq\frac{\pi}{2}\right\}$, and again the imaginary axis from $z=i\varepsilon$ to $z=i\infty$. Let $C_\varepsilon^-(\varepsilon>0)$ be the path which consists of the imaginary axis from $z=-i\infty$ to $z=-i\varepsilon$, the semicircle

$$c_\varepsilon^-=\left\{z:\ z=-\varepsilon e^{i\alpha},\ -\frac{\pi}{2}\leqq\alpha\leqq\frac{\pi}{2}\right\},$$

and again the imaginary axis from $z=i\varepsilon$ to $z=i\infty$. Let $C_\varepsilon^+(R)$ $(0<\varepsilon<R)$ be a path taken in the negative direction and containing C_ε^+ from $z=-iR$ to $z=iR$ and the semicircle $c_R^+=\left\{z:\ z=Re^{-i\alpha},\ -\frac{\pi}{2}\leqq\alpha\leqq\frac{\pi}{2}\right\}$. Let $C_\varepsilon^-(R)$ $(0<\varepsilon<R)$ be a path taken in the positive direction and containing C_ε^- from $z=-iR$ to $z=iR$ and the semicircle $c_R^-=\left\{z:\ z=-Re^{-i\alpha},\ -\frac{\pi}{2}\leqq\alpha\leqq\frac{\pi}{2}\right\}$.

Since $\Phi^+(z)$ is regular inside $C_\varepsilon^+(R)$ and continuous on the boundary, it follows by Cauchy's integral formula (see e.g. [11] p. 112) that

$$\frac{s}{2\pi i}\int_{C_\varepsilon^+(R)}\frac{\Phi^+(z)}{z(s-z)}\,dz=\Phi^+(s)$$

for $0<\varepsilon<\operatorname{Re}(s)$ and $|s|<R$. Since $|\Phi^+(z)|\leqq\|\Phi\|$ for $\operatorname{Re}(z)\geqq 0$, if we let $R\to\infty$ the integral on the semicircle c_R^+ tends to 0. Hence we obtain that

$$\frac{s}{2\pi i}\int_{C_\varepsilon^+}\frac{\Phi^+(z)}{z(s-z)}\,dz=\Phi^+(s) \tag{18}$$

for $0<\varepsilon<\mathrm{Re}(s)$. If $\varepsilon\to 0$, then in (18) the integral taken along the semicircle c_ε^+ tends to $\Phi^+(0)/2=\Phi(0)/2$ and thus by (18)

$$\lim_{\varepsilon\to 0}\frac{s}{2\pi i}\int_{L_\varepsilon}\frac{\Phi^+(z)}{z(s-z)}dz+\frac{1}{2}\Phi(0)=\Phi^+(s) \tag{19}$$

for $\mathrm{Re}(s)>0$.

Next we observe that

$$\Phi(s)-\Phi^+(s)=\mathbf{E}\{\zeta e^{s[-\eta]^+}\}-\Phi(0) \tag{20}$$

for $\mathrm{Re}(s)=0$. This follows from the identity $e^{-s\eta}-e^{-s\eta^+}\equiv e^{-s\eta^+}(e^{s[-\eta]^+}-1)\equiv$ $\equiv e^{s[-\eta]^+}-1$. If we extend the definition of $\Phi(s)-\Phi^+(s)$ for $\mathrm{Re}(s)\leqq 0$ by (20), then $\Phi(s)-\Phi^+(s)$ becomes regular in the domain $\mathrm{Re}(s)<0$ and continuous for $\mathrm{Re}(s)\leqq 0$. Obviously, $|\Phi(s)-\Phi^+(s)|\leqq 2\|\Phi\|$ for $\mathrm{Re}(s)\leqq 0$. By Cauchy's integral theorem (see e.g. [11] p. 105) it follows that

$$\frac{s}{2\pi i}\int_{C_\varepsilon^-(R)}\frac{\Phi(z)-\Phi^+(z)}{z(s-z)}dz=0$$

for $\mathrm{Re}(s)>0$. If we let $R\to\infty$, we obtain that

$$\frac{s}{2\pi i}\int_{C_\varepsilon^-}\frac{\Phi(z)-\Phi^+(z)}{z(s-z)}dz=0. \tag{21}$$

If $\varepsilon\to 0$, the part of the integral taken along the semicircle of radius ε tends to $[\Phi^+(0)-\Phi(0)]/2=0$, and thus by (21)

$$\lim_{\varepsilon\to 0}\frac{s}{2\pi i}\int_{L_\varepsilon}\frac{\Phi(z)-\Phi^+(z)}{z(s-z)}dz=0. \tag{22}$$

If we add (19) and (22), we obtain (17) which was to be proved. For $\mathrm{Re}(s)=0$ the function $\Phi^+(s)$ can be obtained by continuity or by an integral representation similar to (17).

We note that if $\Phi(s)=\mathbf{E}\{\zeta e^{-s\eta}\}$ exists for some $\varepsilon>0$, that is, if $\mathbf{E}\{|\zeta e^{-\varepsilon\eta}|\}<\infty$, then

$$\Phi^+(s)=\frac{s}{2\pi i}\int_{C_\varepsilon^+}\frac{\Phi(z)}{z(s-z)}dz \tag{23}$$

for $\mathrm{Re}(s)>\varepsilon>0$. For in this case (21) remains valid if C_ε^- is replaced by C_ε^+, and hence (23) follows by (18).

6. A factorization. Finally, we show that for $|\varrho|\,\|\gamma\|<1$ we can also obtain $\mathrm{T}\{\log[1-\varrho\gamma(s)]\}$ by another method, namely, by the method of factorization.

Let $\gamma(s)\in\mathbf{R}$, $|\varrho|\,\|\gamma\|<1$ and suppose that

$$1-\varrho\gamma(s) = \Gamma^{+}(s,\varrho)\Gamma^{-}(s,\varrho) \tag{24}$$

for $\mathrm{Re}(s)=0$, where $\Gamma^{+}(s,\varrho)$ and $\Gamma^{-}(s,\varrho)$ as functions of s satisfy the following requirements:

A_1: $\Gamma^{+}(s,\varrho)$ is regular in the domain $\mathrm{Re}(s)>0$,
A_2: $\Gamma^{+}(s,\varrho)$ is continuous and free from zeros in $\mathrm{Re}(s)\geqq 0$,
A_3: $\log\Gamma^{+}(s,\varrho)/s\to 0$ if $\mathrm{Re}(s)\geqq 0$ and $|s|\to\infty$,
B_1: $\Gamma^{-}(s,\varrho)$ is regular in the domain $\mathrm{Re}(s)<0$,
B_2: $\Gamma^{-}(s,\varrho)$ is continuous and free from zeros in $\mathrm{Re}(s)\leqq 0$,
B_3: $\log\Gamma^{-}(s,\varrho)/s\to 0$ if $\mathrm{Re}(s)\leqq 0$ and $|s|\to\infty$.

Such a factorization always exists. For example,

$$\Gamma^{+}(s,\varrho)=e^{\mathrm{T}\{\log[1-\varrho\gamma(s)]\}} \quad\text{and}\quad \Gamma^{-}(s,\varrho)=e^{\log[1-\varrho\gamma(s)]-\mathrm{T}\{\log[1-\varrho\gamma(s)]\}} \tag{25}$$

satisfy all the requirements. Actually, the above requirements determine $\Gamma^{+}(s,\varrho)$ and $\Gamma^{-}(s,\varrho)$ up to a factor depending only on ϱ. This is the content of the next theorem.

Theorem 3. *If* $\gamma(s)\in\mathbf{R}$, $|\varrho|\,\|\gamma\|<1$ *and*

$$1-\varrho\gamma(s) = \Gamma^{+}(s,\varrho)\Gamma^{-}(s,\varrho) \tag{26}$$

for $\mathrm{Re}(s)=0$, *where* $\Gamma^{+}(s,\varrho)$ *and* $\Gamma^{-}(s,\varrho)$ *satisfy the requirements* A_1, A_2, A_3 *and* B_1, B_2, B_3 *respectively, then*

$$\mathrm{T}\{\log[1-\varrho\gamma(s)]\} = \log\Gamma^{+}(s,\varrho)+\log\Gamma^{-}(0,\varrho) \tag{27}$$

for $\mathrm{Re}(s)\geqq 0$.

Proof. We prove (27) for $\mathrm{Re}(s)>0$; the case $\mathrm{Re}(s)=0$ then follows by continuity. Let us define the paths L_ε, C_ε^{+}, C_ε^{-}, $C_\varepsilon^{+}(R)$, $C_\varepsilon^{-}(R)$ in the same way as in the proof of Theorem 2. Then we have

$$\frac{s}{2\pi i}\int\limits_{C_\varepsilon^{+}}\frac{\log\Gamma^{+}(z,\varrho)}{z(s-z)}\,dz = \log\Gamma^{+}(s,\varrho) \tag{28}$$

for $0<\varepsilon<\mathrm{Re}(s)$ and

$$\frac{s}{2\pi i}\int\limits_{C_\varepsilon^{-}}\frac{\log\Gamma^{-}(z,\varrho)}{z(s-z)}\,dz = 0 \tag{29}$$

for Re $(s)>0$, Indeed, (28) and (29) follow in a similar way as (18) and (21): first we integrate along the paths $C_\varepsilon^+(R)$ and $C_\varepsilon^-(R)$, respectively, and then let $R\to\infty$. If $\varepsilon\to 0$ in (28) and (29), then we get

$$\lim_{\varepsilon\to 0}\frac{s}{2\pi i}\int_{L_\varepsilon}\frac{\log\Gamma^+(z,\varrho)}{z(s-z)}\,dz+\frac{1}{2}\log\Gamma^+(0,\varrho)=\log\Gamma^+(s,\varrho) \tag{30}$$

and

$$\lim_{\varepsilon\to 0}\frac{s}{2\pi i}\int_{L_\varepsilon}\frac{\log\Gamma^-(z,\varrho)}{z(s-z)}\,dz-\frac{1}{2}\log\Gamma^-(0,\varrho)=0 \tag{31}$$

for Re $(s)>0$. Adding (30) and (31) we obtain (27) for Re $(s)>0$. This completes the proof of the theorem.

By using (27) we can express (8) also in the form

$$\sum_{n=0}^{\infty}\Gamma_n(s)\varrho^n=\frac{1}{\Gamma^+(s,\varrho)}\mathbf{T}\left\{\frac{\Gamma_0(s)}{\Gamma^-(s,\varrho)}\right\}, \tag{32}$$

where Re $(s)\geqq 0$ and $|\varrho|\,\|\gamma\|<1$. If $\Gamma_0(s)\equiv 1$, then (8) or (32) reduces to

$$\sum_{n=0}^{\infty}\Gamma_n(s)\varrho^n=\frac{1}{\Gamma^+(s,\varrho)\Gamma^-(0,\varrho)}, \tag{33}$$

where Re $(s)\geqq 0$ and $|\varrho|\,\|\gamma\|<1$.

The above results have numerous possible applications in the theory of probability and stochastic processes. Without going into details, we mention only the solution of the problem formulated in the Introduction. If we denote by $\gamma(s)$ the Laplace—Stieltjes transform of $\mathbf{P}\{\xi_n\leqq x\}$, that is, $\gamma(s)=\mathbf{E}\{e^{-s\xi_n}\}$ for Re $(s)=0$ and $n=1,2,\dots$, and by $\Gamma_n(s)$ the Laplace—Stieltjes transform of $\mathbf{P}\{\eta_n\leqq x\}$, that is, $\Gamma_n(s)=\mathbf{E}\{e^{-s\eta_n}\}$ for Re $(s)\geqq 0$ and $n=0,1,2,\dots$, then the generating function of the sequence $\{\Gamma_n(s)\}$ is given by (8) or by (32) for $|\varrho|<1$. If, in particular, $\mathbf{P}\{\eta_0=0\}=1$, that is, $\Gamma_0(s)\equiv 1$, then

$$\sum_{n=0}^{\infty}\Gamma_n(s)\varrho^n=e^{-\mathbf{T}\{\log[1-\varrho\gamma(s)]\}}=\exp\left\{\sum_{n=1}^{\infty}\frac{\varrho^n}{n}\mathbf{T}\{[\gamma(s)]^n\}\right\} \tag{34}$$

for $|\varrho|<1$ and Re $(s)\geqq 0$. The first version of (34) is the general case of a formula of F. Pollaczek [12] and the second version can be reduced to a formula of F. Spitzer [16].

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Uniform embedding of a metric space in Hilbert space

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It is known that every regular space having a σ-discrete base (or, what is the same, having a σ-locally finite base) can be homeomorphically embedded in a Hilbert space (SMIRNOV [1]). In the following we prove that if, in addition, in a regular space a particular metric is given, the embedding can be chosen uniformly continuous. Thus we shall prove the following

Theorem. *Every metric space X can be embedded in a Hilbert space by a uniformly continuous homeomorphism.*

Proof. First we give a definition: A collection of subsets of a space is called *discrete* if every point of the space has a neighbourhood which intersects at most one element of the collection. The collection is called *σ-discrete* if it can be decomposed into an at most countable infinity of discrete subcollections.

The proof of the theorem will be based on the result of BING [2] according to which *every metrizable space has a σ-discrete base.*

Thus, let $\mathscr{B}=\bigcup\limits_{1}^{\infty}\mathscr{B}_n$ be a σ-discrete base of X, where the subcollections $\mathscr{B}_n$ $(n=1, 2, \dots)$ are discrete. We may also assume that every element of $\mathscr{B}$ has a diameter less than 1. For fixed $U\in\mathscr{B}$ and natural number j we consider the set

$$U^j=\left\{x: \varrho(x, \overline{U})>\frac{1}{j}\right\},$$

where $\overline{U}$ denotes the complement of U, and the function

$$f_{(U,j)}(x)=\frac{\varrho(x, \overline{U})}{\varrho(x, \overline{U})+\varrho(x, U^j)}$$

(ϱ denotes the metric of X). Thus we have $0\leqq f_{(U,j)}(x)\leqq 1$; and as for fixed $x\in X$ and for every natural number i there exists at most one $U\in\mathscr{B}_i$ with $x\in U$, it follows that $f_{(U,j)}(x)=0$ for every pair (U,j) with the possible exception of at most a countable infinity of pairs (U,j).

Let $k(i,j)$ be a 1—1 mapping which maps the set of all pairs onto the set of natural numbers and define

$$g_{(U,j)}(x)=f_{(U,j)}(x)/k(i,j)$$

for $U\in\mathscr{B}_i$. Let H be a Hilbert space the dimension of which is equal to the cardinality of all possible pairs (U,j) and for any $x\in X$ consider the function

$$G(x)=\{g_{(U,j)}(x)\}_{(U,j)}$$

which evidently maps X into H. We show that $G(x)$ is a homeomorphism which maps X into H in a uniformly continuous manner. To do this first we remark that both $\varrho(x,\overline{U})$ and $\varrho(x,U^j)$ fulfil a Lipschitz condition with constant 1, further $\varrho(x,\overline{U})\leqq 1$ and $\varrho(x,\overline{U})+\varrho(x,U^j)\geqq\frac{1}{j}$. Thus a simple calculation shows that if $\varrho(x,y)\leqq d$ then

$$|f_{(U,j)}(x)-f_{(U,j)}(y)|\leqq(2j^2+j)d.$$

But for fixed $x, y\in X$ and for every pair i, j there are among the numbers $\{f_{(U,j)}(x), f_{(U,j)}(y)\}_{U\in\mathscr{B}_i}$ at most two different from zero, thus we get for any natural number n

$$\|G(x)-G(y)\|^2=\sum_{(U,j)}|g_{(U,j)}(x)-g_{(U,j)}(y)|^2\leqq 2d^2\sum_{k(i,j)\leqq n}\frac{(2j^2+j)^2}{k^2(i,j)}+2\sum_{m=n}^{\infty}\frac{2}{m^2}.$$

Let $\varepsilon>0$ be given and choose n so large that $\sum_{m=n}^{\infty}\frac{1}{m^2}<\frac{\varepsilon^2}{4}$ and d so small that

$$d^2\sum_{k(i,j)\leqq n}\frac{(2j^2+j)^2}{k^2(i,j)}<\frac{\varepsilon^2}{4};$$

then $\|G(x)-G(y)\|<\varepsilon$, which gives the uniform continuity of $G(x)$.

On the other hand if, $x\neq y$ then there exist U and i such that $x\in U\in\mathscr{B}_i$ and $y\overline{\in}U$. If j is large enough then also $x\in U^j$. But in this case $f_{(U,j)}(x)=1$ and $f_{(U,j)}(y)=0$ which shows that G^{-1} exists.

The continuity of G^{-1} can be proved in the following manner. Let V be an arbitrary neighbourhood of $x\in X$. Then there exists i, j and $U\subset V$ with $x\in U^j\subset\subset U\in\mathscr{B}_i$; now if $\|G(x)-G(y)\|<\frac{1}{k(ij)}$, then also $|f_{(U,j)}(x)-f_{(U,j)}(y)|<1$, from which we get $y\in U^j\subset V$.

Remark. The construction shows that the dimension of H depends only upon the cardinality of a base of X or, what is the same, upon the minimal cardinality of a dense subset of X.

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On n-permutable equational classes

By E. T. SCHMIDT in Budapest

The product $\Theta \circ \Phi$ of two congruences Θ, Φ of an algebra A is defined by the following rule: $a \equiv b(\Theta \circ \Phi)$ if and only if $c \in A$ exists such that $a \equiv c(\Theta)$ and $c \equiv b(\Phi)$. Two congruences Θ_1 and Θ_2 are n-permutable if and only if $\Theta_1 \circ \Theta_2 \circ \circ \Theta_1 \circ \Theta_2 \circ \cdots = \Theta_2 \circ \Theta_1 \circ \Theta_2 \circ \Theta_1 \circ \cdots$, where on both sides there are n factors. An algebra A is n-permutable if every two congruences in A are n-permutable. We define an equational class to be n-permutable if every algebra of this class is n-permutable. It is well known, that an n-permutable equational class is $(n+1)$-permutable. In [1] G. GRÄTZER asks for examples of equational classes which show that n-permutability and $(n+1)$-permutability are not equivalent[1]). In this note we give an example with this property.

Theorem. *For every natural number $n > 2$ there exists an $(n+1)$-permutable equational class $\mathcal{K}_n$ which is not n-permutable.*

Proof. Let n be a natural number. An n-Boolean algebra

$$\mathcal{B} = (B;\ \vee, \wedge, f_1(x), \ldots, f_n(x), o_0, o_1, \ldots, o_n)$$

is an algebra with two binary operations $\vee$, $\wedge$, n unary operations $f_1(x), \ldots, f_n(x)$ and $n+1$ nullary operations $o_0, o_1, \ldots, o_n$, such that the following conditions are satisfied:

1. $(B;\ \vee, \wedge)$ is a distributive lattice;

2. $x \vee o_n = o_n$; $x \vee o_0 = x$ for all $x \in B$;

3. $[(x \vee o_{i-1}) \wedge o_i] \vee f_i(x) = o_i$; $[(x \vee o_{i-1}) \wedge o_i] \wedge f_i(x) = o_{i-1}$.

The class of all n-Boolean algebras is denoted by $\mathcal{K}_n$. If $o_{i-1} \leqq x \leqq o_i$ then $f_i(x)$ is the relative complement from x in $[o_{i-1}, o_i]$, i.e. this interval is a Boolean lattice. A 1-Boolean algebra is a Boolean algebra. A finite chain $\mathcal{C}_n$ of $n+1$ elements is

[1]) For $n=2$ A. MITSCHKE [2] has solved this problem.

an n-Boolean algebra, if we take its elements as nullary operations: $o_0 < o_1 < o_2 < \cdots$ $\cdots < o_n$ $(o_i \in \mathscr{C}_n)$, and $f_i(x) = o_i$ if $x < o_i$, $f_i(x) = o_{i-1}$ *if* $x \geqq o_i$. The congruences of $\mathscr{C}_n$ are the lattice-congruences, i.e. $\mathscr{C}_n$ is not n-permutable. This shows that $\mathscr{K}_n$ is not n-permutable.

Let B denote an arbitary n-Boolean algebra and $x, y \in B$, $x > y$. Set $a_i = (o_i \wedge x) \vee y$. (Then is $a_0 = y$, $a_n = x$.) If Θ_1 and Θ_2 are arbitary congruences from B, such that $x \equiv y$ $(\Theta_1 \vee \Theta_2)$, then $a_{i-1} \equiv a_i (\Theta_1 \vee \Theta_2)$ $(i = 1, 2, \ldots, n)$. The interval $[a_{i-1}, a_i]$ is projective to a subinterval of $[o_{i-1}, o_i]$, i.e. $[a_{i-1}, a_i]$ is a Boolean lattice. Every Boolean lattice is 2-permutable and so for every i $(i = 1, 2, \ldots, n)$ there exists a $t_i \in [a_{i-1}, a_i]$ such that

$$a_{i-1} \equiv t_i(\Theta_1) \ i \text{ odd}, \quad a_{i-1} \equiv t_i(\Theta_2) \ i \text{ even}, \quad a_i \equiv t_i(\Theta_1) \ i \text{ even}, \ a_i \equiv t_i(\Theta_2) \ i \text{ odd}.$$

We have therefore between x, y a chain $y_0 = a_0 = y$, $y_1 = t_1$, $y_2 = t_2, \ldots,$ $y_n = x = a_n$ with $n+1$ elements, such that $y_{i-1} \equiv y_i(\Theta_1)$ if i even and $y_{i-1} \equiv y_i(\Theta_2)$ if i odd. $\mathscr{K}_n$ is therefore $(n+1)$-permutable.

Remark. An equational class is $(n+1)$-permutable if and only if there exists $(n+2)$-ary algebraic operations $p_0, \ldots, p_{n+1}$ satisfying the following identities (see [3]):

$$p_0(x_0, \ldots, x_{n+1}) = x_0, \ p_{i-1}(x_0, x_0, x_2, x_2, \ldots) = p_i(x_0, x_0, x_2, x_2, \ldots) \ (i = \text{even}),$$

$$p_{i-1}(x_0, x_1, x_1, x_3, x_3, \ldots) = p_i(x_0, x_1, x_1, x_3, x_3, \ldots) \ (i \text{ odd}),$$

$$p_{n+1}(x_0, \ldots, x_{n+1}) = x_{n+1}.$$

A. Mitschke and H. Werner have considered for the class $\mathscr{K}_n$ the algebraic operations:

$$p_i(x_0, x_1, \ldots, x_{n+1}) = (x_i \wedge f_{n+1-i}(x_{i+1}) \vee x_{i+2}) \vee (x_{i+2} \wedge (f_i(x_{i+1}) \vee x_i))$$

which show that $\mathscr{K}_n$ is $(n+1)$-permutable.

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Basic study of general products and homogeneous homomorphisms. I

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§ 1. Intruduction

We frequently meet the problem to study semigroups S which are homomorphic onto a given semigroup T. Of course the problem in this form is too vague to be treated in general. Let us restrict ourselves to the following problem:

Given a semigroup T, study semigroups S such that S is homomorphic onto T under a map f and such that the cardinal number of the inverse image set of each element of T is constant, i.e. given m

$$|tf^{-1}|=m \quad \text{for all} \quad t \in T.$$

Such a homomorphism of S is called a *homogeneous homomorphism*. Let A be a set with cardinality m. We will introduce a concept "general product" of a set A by a semigroup T, which will be equivalent to the concept of homogeneous homomorphism. This concept includes the various known concepts. Then the first problem proposed above will be connected with the second restricted problem, that is, if S is homomorphic to T then the homomorphism can be extended to a homogeneous homomorphism of certain semigroup S' to T. Related to general product, we will consider the system of all binary operations defined on a set.

A part of the outline of this paper was reported in [11], [12] without proof. This paper is to report basic results of general products but its development and applications will be reported as the continuation in the future. Computational results related to this paper will be separately reported though a part of those were done in [12].

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§ 2. The system of operations

Let E be a set and $\mathscr{B}_E$ be the set of all binary operations defined on E. Let $x, y \in E$, $\theta \in \mathscr{B}_E$ and let $x\theta y$ denote the product of x and y by θ. A groupoid with θ defined on E is denoted by $E(\theta)$. The equality of elements $\theta, \eta \in \mathscr{B}_E$ is defined as follows:

$$\theta=\eta \quad \text{if and only if} \quad x\theta y=x\eta y \quad \text{for all} \quad x, y \in E.$$

Let $a \in E$ be fixed. Two binary operati ons ${}_a\Pi$ and Π_a are defined in $\mathscr{B}_E$ as follows:

(1) $\quad x(\theta_a\Pi\eta)y=(x\theta a)\eta y \quad$ and $\quad x(\theta\Pi_a\eta)y=x\theta(a\eta y) \quad$ for $\quad \theta, \eta \in \mathscr{B}_E, \ x, y \in E.$

It is clear that θ is associative if and only if $\theta a^*\theta=\theta^* a\theta$ for all $a \in E$.

Proposition 1. *$\mathscr{B}_E$ is a semigroup with respect to ${}_a\Pi$ and Π_a for all $a \in E$.*

Proof. *To prove* $(\xi_a\Pi\eta)a^*\theta=\xi_a\Pi(\eta_a\Pi\theta)$, $\xi, \eta, \theta \in \mathscr{B}_E$. We have

$$x[(\xi_a\Pi\eta)_a\Pi\theta]y=\{x(\xi_a\Pi\eta)a\}\theta y=\{(x\xi a)\eta a\}\theta y=$$

$$=(x\xi a)(\eta_a\Pi\theta)y=x[\xi_a\Pi(\eta_a\Pi\theta)]y \quad \text{for all} \quad x, y \in E.$$

Likewise we can prove $(\xi\Pi_a\eta)\Pi_a\theta=\xi\Pi_a(\eta^* a\theta)$.

The semigroups $\mathscr{B}_E$ with ${}_a\Pi$ and Π_a are denoted by $\mathscr{B}_E({}_a\Pi)$ or $\mathscr{B}({}_a\Pi)$, and $\mathscr{B}_E(\Pi_a)$ or $\mathscr{B}(\Pi_a)$ respectively.

Let φ be a permutation of E. For $\theta \in \mathscr{B}_E$, $\theta\varphi$ is defined as follows:

(2) $$x(\theta\varphi)y=[(x\varphi^{-1})\theta(y\varphi^{-1})]\varphi$$

or, by substituting x for $x\varphi^{-1}$,

(3) $$(x\theta y)\varphi=(x\varphi)(\theta\varphi)(y\varphi).$$

The mapping $\theta \to \theta\varphi$ is a permutation of $\mathscr{B}_E$. For any $\eta \in \mathscr{B}_E$, define θ by

$$x\theta y=[(x\varphi)\eta(y\varphi)]\varphi^{-1}.$$

Then we can easily prove $\theta\varphi=\eta$. Hence the mapping $\theta \to \theta\varphi$ is onto. To prove one-to-one. Suppose $\theta\varphi=\eta\varphi$. Then

$$[(x\varphi^{-1})\theta(y\varphi^{-1})]\varphi=[(x\varphi^{-1})\eta(y\varphi^{-1})]\varphi.$$

Since φ is a permutation of E, we have $(x\varphi^{-1})\theta(y\varphi^{-1})=(x\varphi^{-1})\eta(y\varphi^{-1})$ where $x\varphi^{-1}$, $y\varphi^{-1}$ run throughout E and hence $\theta=\eta$.

Thus φ induces a permutation of $\mathscr{B}_E$. This permutation is still denoted by φ.

Proposition 2. $(\theta_a\Pi\eta)\varphi=(\theta\varphi)_{a\varphi}\Pi(\eta\varphi)$ *and* $(\theta\Pi_a\eta)\varphi=(\theta\varphi)\Pi_{a\varphi}(\eta\varphi)$.

Proof. we have

$$x(\theta_a\Pi\eta)\varphi y=[x\varphi^{-1}(\theta_a\Pi\eta)y\varphi^{-1}]\varphi=[\{(x\varphi^{-1})\theta a\}\eta(y\varphi^{-1})]\varphi=$$

$$=[x(\theta\varphi)(a\varphi)](\eta\varphi)y=x[(\theta\varphi)_{a\varphi}\Pi(\eta\varphi)]y \quad \text{for} \quad x, y\in E.$$

For $\theta\in\mathscr{B}_E$ we define θ' as follows:

(4) $$x\theta' y=y\theta x.$$

Then we have

Proposition 3. $(\theta_a\Pi\eta)'=\eta'\Pi_a\theta'$.

Proof. For all $x, y\in E$, $x(\theta_a\Pi\eta)'y=y(\theta_a\Pi\eta)x=(y\theta a)\eta x=xy'(y\theta a)=x\eta'(a\theta' y)=$ $=x(\eta'\Pi_a\theta')y$.

Proposition 4. *$\mathscr{B}({}_a\Pi)\cong\mathscr{B}({}_b\Pi)$ and $\mathscr{B}(\Pi_a)\cong\mathscr{B}(\Pi_b)$ for all $a, b\in E$. Furthermore, $\mathscr{B}({}_a\Pi)$ is anti-isomorphic with $\mathscr{B}(\Pi_a)$.*

Proof. Let φ be a permutation of E such that $a\varphi=b$. By Proposition 2, $(\theta_a\Pi\eta)\varphi=(\theta\varphi)_b\Pi(\eta\varphi)$. This shows that φ is an isomorphism of $\mathscr{B}({}_a\Pi)$ onto $\mathscr{B}({}_b\Pi)$. Similarly we have by the second part of Proposition 2 that φ is an isomorphism of $\mathscr{B}(\Pi_a)$ onto $\mathscr{B}(\Pi_b)$.

§ 3. General product of a set by a groupoid

Let S be a set and T be a groupoid. Consider a mapping Θ of $T\times T$ into $\mathscr{B}_S$:

$$(\alpha, \beta)\Theta=\theta_{\alpha,\beta}, \qquad (\alpha, \beta)\in T\times T.$$

Now $S\times T = \{(x, \alpha);\ x\in S,\ \alpha\in T\}$ in which $(x, \alpha)=(y, \beta)$ if and only if $x=y$, $\alpha=\beta$. Given S, T, Θ, a binary operation is defined on $S\times T$ as follows:

(5) $$(x, \alpha)(y, \beta)=(x\theta_{\alpha,\beta}y, \alpha\beta).$$

Definition. The groupoid $S\times T$ with (5) is called a *general product* of a set S by a groupoid T with respect to Θ, and is denoted by $S\overline{\times}_\Theta T$. If it is not necessary to specify Θ, it is denoted by $S\overline{\times} T$.

Definition. If a groupoid D is isomorphic onto some $S\overline{\times}_\Theta T$, $|S|>1$, $|T|>1$, then D is called *general-product decomposable (gp-decomposable)*.

Immediately we see that $S\overline{\times}_\Theta T$ is homomorphic onto T by the mapping $p:(x, \alpha)\to\alpha$. This mapping is called the projection of $S\overline{\times} T$ onto T. Likewise we can define the projection of $S\overline{\times} T$ onto S.

Proposition 5. *$S\bar{\times}_\Theta T$ is a semigroup if and only if T is a semigroup and*

$$\theta_{\alpha,\beta}{}_a\Pi\theta_{\alpha\beta,\gamma}=\theta_{\alpha,\beta\gamma}\Pi_a\theta_{\beta,\gamma} \quad \textit{for all} \quad a\in S, \quad \textit{all} \quad \alpha,\beta,\gamma\in T.$$

Proof. The proposition is immediately proved as follows:

$$[(x,\alpha)(y,\beta)](z,\gamma)=(x\theta_{\alpha,\beta}y,\alpha\beta)(z,\gamma)=((x\theta_{\alpha,\beta}y)\theta_{\alpha\beta,\gamma}z,(\alpha\beta)\gamma),$$
$$(x,\alpha)[(y,\beta)(z,y)]=(x,\alpha)(y\theta_{\beta,\gamma}z,\beta\gamma)=(x\theta_{\alpha,\beta\gamma}(y\theta_{\beta,\gamma}z),\alpha(\beta\gamma)).$$

Definition. Let g be a homomorphism of a groupoid D onto a groupoid T: $D=\bigcup_{\alpha\in T}D_\alpha$; $D_\alpha g=\alpha$. If either $|D_\alpha|=1$ for all α or if $|T|=1$, g is called *trivial*; otherwise g is called *proper*. If $|D_\alpha|=|D_\beta|$ for all $\alpha,\beta\in T$, then g is called a *homogeneous homomorphism* (h-homomorphism) of D, or D is said to be *homogeneously homomorphic* (h-homomorphic) onto T.

Theorem 6. *A groupoid D is isomorphic onto $S\bar{\times}_\Theta T$ for some S and some Θ if and only if D is h-homomorphic onto T. More precisely, D is gp-decomposable if and only if D has a proper h-homomorphism.*

Proof. Suppose that D is homogeneously homomorphic onto T under g:

$$D=\bigcup_{\alpha\in T}D_\alpha,\quad D_\alpha g=\alpha.$$

Let S be a set with $|S|=|D_\alpha|$ for all $\alpha\in T$, and f_α be a one-to-one mapping of D_α onto S. After fixing a system $\{f_\alpha;\ \alpha\in T\}$, for each $(\alpha,\beta)\in T\times T$ we define a binary operation $\theta_{\alpha,\beta}$ on S as follows: Let

$$x\theta_{\alpha,\beta}y=[(xf_\alpha^{-1})(yf_\beta^{-1})]f_{\alpha\beta} \tag{6}$$

$x,y\in S$, where $\alpha\beta$ is the product in T. Now

$$D=\bigcup_{\alpha\in T}D_\alpha \quad \text{where} \quad D_\alpha=\{x\in D;\ xg=\alpha\}.$$

Let a be any element of D, hence $a\in D_\alpha$ for some $\alpha\in T$. We define a mapping ψ of D onto $S\times T$ as follows: $a\psi=(af_\alpha,\alpha)$. Then ψ is one-to-one: suppose $(af_\alpha,\alpha)=$ $=(bf_\beta,\beta)$. By the definition of equality we have $\alpha=\beta$, $af_\alpha=bf_\beta$. Since f_α is one-to-one, $a=b$. It is clear that ψ is onto. We shall prove $(ab)\psi=(a\psi)(b\psi)$. Let $a\in D_\alpha$, $b\in D_\beta$. By (6)

$$(ab)\psi=((ab)f_{\alpha\beta},\alpha\beta)=((af_\alpha)\theta_{\alpha,\beta}(bf_\beta),\alpha\beta)=(af_\alpha,\alpha)(bf_\beta,\beta)=(a\psi)(b\psi).$$

Consequently $D\cong S\bar{\times}_\Theta T$.

Conversely, suppose D is isomorphic with $S\bar{\times}_\Theta T$ under a mapping $f: D\to S\bar{\times}_\Theta T$. Let p be the projection of $S\bar{\times}_\Theta T$ onto T:

$$(x,\alpha)\xrightarrow{p}\alpha.$$

Then fp is a homomorphism of D onto T. Let $D_\alpha = \{a \in D;\ a(fp) = \alpha\}$, $D'_\alpha = \{(x, \alpha);\ x \in S\}$. Since f is one-to-one, $|D_\alpha| = |D'_\alpha| = |S|$ for all $\alpha \in T$. This completes the proof of the theorem.

As seen in the proof of Theorem 6, even if D, S, T are given, Θ depends on the choice of $\{f_\alpha; \alpha \in T\}$. Suppose that for given D, S, T,

$$\Theta: \{\theta_{\alpha,\beta};\ (\alpha, \beta) \in T \times T\} \text{ is determined by } \{f_\alpha;\ \alpha \in T\},$$

$$\Theta': \{\theta'_{\alpha,\beta};\ (\alpha, \beta) \in T \times T\} \text{ is determined by } \{f'_\alpha;\ \alpha \in T\}.$$

What relationship is there between Θ and Θ'?

To state the problem generally we need to introduce some terminology:

Definition. Let g and g' be homomorphisms of groupoids A and B onto a groupoid C. An isomorphism h of A into (onto) B is called a restricted isomorphism of A into (onto) B with respect to g and g' or A is restrictedly isomorphic into (onto) B with respect to g and g' if there is a permutation k of C such that the following diagram is commutative:

$$h \cdot g' = g \cdot k \qquad \begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ B & \xrightarrow[g']{} & C \end{array}$$

The permutation k of C is an automorphism of C. Let $\alpha, \beta \in C$. $\alpha = xg$, $\beta = yg$ for some $x, y \in A$. We have

$$(\alpha\beta)k = [(xg)(yg)]k = [(xy)g]k = [(xy)h]g' = (xhg')(yhg') = [(xg)k][(yg)k] = (\alpha k)(\beta k).$$

Now the problem is this: Given S and T, let $D = S \overline{\times}_\Theta T$, $D' = S \overline{\times}_{\Theta'} T$. Let p and p' be the projections of D and D' onto T respectively. Under what condition on Θ and Θ' is D restrictedly isomorphic onto D' with respect to p and p'?

Definition. Let $G(\theta)$ and $G'(\theta')$ be groupoids with binary operations θ, θ' respectively. If there are three one-to-one mappings h, q, r of $G(\theta)$ onto $G'(\theta')$ such that $(x\theta y)r = (xh)\theta'(yq)$ for all $x, y \in G(\theta)$, then we say that $G(\theta)$ is *isotopic* to $G'(\theta')$ (see [1]). If it is necessary to specify h, q, r, we say $G(\theta)$ is (h, q, r)-isotopic to $G'(\theta')$. We denote it by

$$G(\theta) \underset{(h,q,r)}{\approx} G'(\theta') \quad \text{or} \quad G(\theta) \approx G'(\theta').$$

Theorem 7. *Let S and T be fixed. Let $(\alpha, \beta)\Theta = \theta_{\alpha,\beta}$, $(\alpha, \beta)\Theta' = \theta_{\alpha,\beta}$, $\alpha, \beta \in T$. $S \overline{\times}_\Theta T$ is restrictedly isomorphic with $S \overline{\times}_{\Theta'} T$ with respect to p, p' if and only if there is an automorphism $\alpha \to \alpha'$ of T and there is a system $\{f_\alpha;\ \alpha \in T\}$ of permutations of S such that $S(\theta_{\alpha,\beta})$ is $(f_\alpha, f_\beta, f_{\alpha\beta})$-isotopic to $S(\theta'_{\alpha',\beta'})$ for all $\alpha, \beta \in T$.*

Proof. Let h be a restricted isomorphism $S\times_\theta T \to S\times_{\theta'} T$ with respect to the projections p, p' as follows:

$$\begin{tikzcd} S\overline{\times}_\theta T \arrow[r, "p"] \arrow[d, "h"'] & T \arrow[d, "k"] \\ S\overline{\times}_{\theta'} T \arrow[r, "p'"'] & T \end{tikzcd}$$

Let $(x, \alpha)\in S\overline{\times}_\theta T$ and $(x, \alpha)h=(x', \alpha')\in S\overline{\times}_{\theta'}T$. Immediately $\alpha'=\alpha k$, further $(x, \alpha)h=(y, \beta)h$ implies $x=y$ and $\alpha=\beta$. Thus $x\to x'$ is a permutation of S depending on α. This permutation is denoted by l_α and then $(x, \alpha)h=(xl_\alpha, \alpha')$, where $\alpha'=\alpha k$. By using this notation,

$$[(x, \alpha)(y, \beta)]h=(x\theta_{\alpha,\beta}y, \alpha\beta)h=((x\theta_{\alpha,\beta}y)l_{\alpha\beta}, (\alpha\beta)'),$$

$$(x, \alpha)h\cdot(y, \beta)h = (xl_\alpha, \alpha k)(yl_\beta, \beta k) = ((xl_\alpha)\theta'_{\alpha',\beta'}(yl_\beta), \alpha'\beta')$$

and we have $(\alpha\beta)'=\alpha'\beta'$, $(x\theta_{\alpha,\beta}y)l_{\alpha\beta}=(xl_\alpha)\theta'_{\alpha',\beta'}(yl_\beta)$. Therefore

(7) $$S(\theta_{\alpha,\beta}) \underset{(l_\alpha,\, l_\beta,\, l_{\alpha\beta})}{\approx} S(\theta'_{\alpha',\beta'}) \quad \text{for all} \quad \alpha, \beta\in T.$$

Conversely suppose there is an automorphism $k:\alpha\to\alpha'$ of T and a system $\{l_\alpha;\ \alpha\in T\}$ of permutations of S satisfying (7). We define a mapping h of $S\overline{\times}_\theta T$ onto $S\overline{\times}_{\theta'}T$ as follows: $(x, \alpha)h=(xl_\alpha, \alpha')$. Then we can easily see that h is one-to-one and $[(x, \alpha)(y, \beta)]h = (x, \alpha)h\cdot(y, \beta)h$. To prove that h is a restricted isomorphism, observe that $(x, \alpha)hp'=(xl_\alpha, \alpha')p'=\alpha'$ and $(x, \alpha)pk=\alpha k=\alpha'$ for all $(x, \alpha)\in S\overline{\times}_\theta T$; hence $hp'=pk$. Thus the proof of the theorem is completed.

As usual the product $\varrho\cdot\sigma$ of binary relations ϱ, σ on D is defined by

$$\varrho\cdot\sigma = \{(x, y);\ (x, z)\in\varrho, (z, y)\in\sigma \quad \text{for some} \quad z\in D\}.$$

Let $\omega = D\times D$, $\iota=\{(x, x);\ x\in D\}$.

The following theorem characterizes general product in terms of relations.

Theorem 8. *A groupoid D is gp-decomposable if and only if there is a congruence ϱ on D and an equivalence σ on D such that $\varrho\neq\omega$, $\sigma\neq\omega$,*

(8) $$\varrho\cdot\sigma = \omega, \quad \textit{and}$$

(9) $$\varrho\cap\sigma=\iota,$$

in which (8) *can be replaced by $\sigma\cdot\varrho = \omega$. Then $D \cong (D/\sigma)\overline{\times}(D/\varrho)$.*

Proof. Suppose $D \cong S\overline{\times}_\theta T$. Let ϱ be the congruence induced by the homomorphism $g:D\to T$. As stated in the proof of Theorem 1, $D=\bigcup_{\alpha\in T} D_\alpha$ where $|D_\alpha|=|S|$.

Let f_α be a one-to-one mapping of D_α onto S. Now we define a relation σ on D as follows: $x\sigma y$ if and only if $xf_\alpha = yf_\beta$ for α and β such that $x \in D_\alpha$, $y \in D_\beta$.

Now take $a, b \in D$ arbitrarily and assume $a \in D_\alpha$, $b \in D_\beta$. Let $c = bf_\beta f_\alpha^{-1}$. Then $c \in D_\alpha$, and $a\varrho c$, $c\sigma b$. Thus we have proved $\varrho \cdot \sigma = \omega$. Suppose $a\varrho b$ and $a\sigma b$, that is, $a, b \in D_\alpha$ and $af_\alpha = bf_\alpha$. Since f_α is one-to-one, $a = b$; therefore $\varrho \cap \sigma = \iota$.

Conversely, suppose that there is a congruence ϱ, $\varrho \neq \omega$, and an equivalence σ, $\sigma \neq \omega$, on D such that (8) and (9) are satisfied. Let T be the factor groupoid D/ϱ and S be the factor set D/σ. Let A and B be any ϱ-class and σ-class respectively and let $x \in A$, $y \in B$. By (8) there is $z \in D$ such that $x\varrho z$ and $z\sigma y$. This means that $A \cap B \neq \emptyset$. Suppose $x\varrho z$, $z\sigma y$, $x\varrho z'$ and $z'\sigma y$. Then $z\varrho z'$ and $z\sigma z'$. By (9), we have $z = z'$. Thus $A \cap B$ consists of exactly one element. Therefore the cardinal number of each ϱ-class is equal. By Theorem 6, we have $D \cong S \overline{\times} T$.

§ 4. Examples

The following well-known concepts are regarded as examples of general product.

Example 1. *Direct Product.* Suppose Θ maps (α, β) to a constant element θ, that is, $(\alpha, \beta)\Theta = \theta$ for all $\alpha, \beta \in T$. Then θ is automatically associative by Proposition 5. In other words S is a semigroup with θ. Thus $S \overline{\times}_\Theta T$ is the direct product of S and T.

Example 2. *Semi-direct product* (see [3], [6], [7]). Let S and T be semigroups, and Y be a homomorphism of T into the endomorphism semigroup of S, $t \xrightarrow{Y} Y_t$. The semi-direct product of S by T with respect to Y is the set $S \times T$ with the operation

$$(s_1, t_1)(s_2, t_2) = (s_1(Y_{t_1}(s_2)),\ t_1 t_2).$$

This is regarded as $S \overline{\times}_\Theta T$ in which $s_1 \theta_{t_1, t_2} s_2 = s_1 \cdot Y_{t_1}(s_2)$.

Example 3. *Rees' regular representation of completely simple semigroups* (see [2]). Let G be a group and F be a rectangular band

$$F = \{(\lambda, \mu);\ \lambda \in \Lambda,\ \mu \in M\},\quad (\lambda, \mu)(\nu, \xi) = (\lambda, \xi).$$

Let $P = (p_{\mu\lambda})$, $\mu \in M$, $\lambda \in \Lambda$ be a matrix over G. If we define Θ by

$$x\theta_{(\lambda,\mu)(\nu,\xi)} y = x p_{\mu\nu} y,$$

then $G \overline{\times}_\Theta F$ is a completely simple semigroup.

Example 4. *Commutative archimedean cancellative semigroups without idempotent* (see [9]). Such a semigroup is called an $\mathfrak{N}$-semigroup. Let G be an abelian group and N be the set of all non-negative integers. Suppose a function $I: K\times K\to N$ satisfies

(a) $I(\alpha, \beta)=I(\beta, \alpha)$.
(b) $I(\alpha, \beta)+I(\alpha\beta, \gamma) = I(\alpha, \beta\gamma)+I(\beta, \gamma)$.
(c) $I(\varepsilon, \varepsilon)=1$, ε being the identity element of G.
(d) For every $\alpha\in G$ there is a positive integer m such that $I(\alpha^m, \alpha)>0$.

Define an operation on the set $S = N\times G$ by

$$(m, \alpha)(n, \beta) = (m+n+I(\alpha, \beta), \alpha\beta)$$

Then S is an $\mathfrak{N}$-semigroup. Every $\mathfrak{N}$-semigroup is obtained in this manner.

$$S \cong N\overline{\times}_\Theta G \text{ where } m\theta_{\alpha,\beta}n = m+n+I(\alpha, \beta).$$

Example 5. *Group extensions* (see [3], [5]). Let N and H be groups and let G be the group extension of N by H. A mapping $\alpha\to f_\alpha$ associates with each $\alpha\in H$ an automorphism f_α of N such that $f_{\alpha\beta}(x)=f_\alpha f_\beta(x)$, $x\in N$. Consider another mapping, $(\alpha, \beta)\to c_{\alpha,\beta}$ of $H\times H$ into N such that

$$c_{\alpha,\beta}f_{\alpha\beta}(x)c_{\alpha\beta,\gamma}=f_{\alpha\beta}(x)f_\alpha(c_{\beta,\gamma})c_{\alpha,\beta\gamma}$$

for all $\alpha, \beta, \gamma\in H$ and all $x\in N$. Then

$$G\cong N\overline{\times}_\Theta H \quad\text{where}\quad x\theta_{\alpha,\beta}y=xf_\alpha(y)c_{\alpha,\beta}.$$

Example 6. *Schreier Extension.* Let A and B be commutative semigroups with identity element. A Schreier extension of A by B in Hancock's sense [4] or Rédei's sense is an example of general product. Examples 4 and 5 are Schreier extensions.

Example 7. *H-semigroups* (see [10]).

As an extremely special case finite semigroups S having property that all homomorphisms of S are homogeneous were studied.

§ 5. Left (right) general product

Definition. A general product $S\overline{\times}_\Theta T$ is called a *left general product* of S by T if

(10) $$(\alpha, \beta)\Theta=(\alpha, \gamma)\Theta \quad\text{for all}\quad \alpha, \beta, \gamma\in T.$$

$S\overline{\times}_\Theta T$ is called a *right general product* of S by T if

(11) $$(\alpha, \beta)\Theta=(\gamma, \beta)\Theta \quad\text{for all}\quad \alpha, \beta, \gamma\in T.$$

In the case (10) $\theta_{\alpha,\beta}$ depends on only α, so $\theta_{\alpha,\beta}$ is denoted by θ_{α}. Then (5) becomes

$$\theta_{\alpha\cdot a}\Pi\theta_{\alpha\beta\cdot}=\theta_{\alpha\cdot}\Pi_a\theta_{\beta\cdot} \quad \text{for all} \quad \alpha,\beta\in T, \quad \text{all} \quad a\in S. \tag{12}$$

In the case (11), $\theta_{\alpha,\beta}$ is denoted by $\theta_{\cdot\beta}$ and we have

$$\theta_{\cdot\alpha a}\Pi\theta_{\cdot\beta}=\theta_{\cdot\alpha\beta}\Pi_a\theta_{\cdot\beta} \quad \text{for all} \quad \alpha,\beta\in T, \quad \text{all} \quad a\in S. \tag{13}$$

σ By a left congruence we mean a left compatible equivalence, i.e. an equivalence satisfying

$$x\sigma y \Rightarrow zx \;\sigma\; zy \quad \text{for all} \quad z. \tag{14}$$

Theorem 9. *Let D be a groupoid. D is isomorphic with a left general product of a set S, $|S|>1$, by a groupoid T, $|T|>1$, if and only if there is a congruence ϱ on D and a left congruence σ on D such that $\varrho\neq\omega$, $\sigma\neq\omega$, $D/\varrho\cong T$, $|D/\sigma|=|S|$, and $\varrho\cdot\sigma=\omega$, $\varrho\cap\sigma=\iota$.*

Proof. Theorem 8 is applicable to this theorem except for (14). Suppose D is isomorphic with a left general product of S by T under a map h. Let $g=hp$ and $f=hq$

$$\begin{array}{ccccc} D & \xrightarrow{h} & S\overline{\times}T & \xrightarrow{p} & T \\ & & & \searrow^{q} & \\ & & & & S \end{array}$$

Let f_α denote the restriction of f to D_α and let $D_\alpha g=\alpha$, $(x,\alpha)\in S\times T$, $(x,\alpha)p=\alpha$, $(x,\alpha)q=x$; ϱ and σ are defined as in the first part of the proof of Theorem 8. We need prove (14) only. Suppose $a, b, c\in D$ and $b\sigma c$ and let

$$ah=(x,\alpha), \quad bh=(y,\beta), \quad ch=(y,\gamma).$$

Then $(x,\alpha)(y,\beta)=(x\theta_{\alpha\cdot}y,\alpha\beta)$, $(x,\alpha)(y,\gamma)=(x\theta_{\alpha\cdot}y,\alpha\gamma)$. This shows that $(ab)f_{\alpha\beta}=(ac)f_{\alpha\gamma}$ or $ab \;\sigma\; ac$ and we have proved (14).

Conversely suppose that a congruence ϱ and a left congruence σ on D exist. By Theorem 8, D is isomorphic with a general product $D/\sigma\overline{\times}D/\varrho$, ϱ and σ naturally induce relations on $D/\sigma\overline{\times}D/\varrho$. In this sense ϱ and σ can be regarded as the relations on $D/\sigma\overline{\times}D/\varrho$. By the assumption, $(y,\beta)\sigma(y,\gamma)$ implies $(x,\alpha)(y,\beta)\sigma(x,\alpha)(y,\gamma)$, hence $x\theta_{\alpha,\beta}y=x\theta_{\alpha,\gamma}y$ which means that $x\theta_{\alpha,\beta}y$ is independent of β. Thus we have proved that $D/\sigma\overline{\times}D/\varrho$ is a left general product.

§ 6. The structure of $\mathscr{B}_E({}_a\Pi)$.

Let a be a fixed element of E, $|E|>1$, and let $\theta\in\mathscr{B}_E$. We define f_θ and g_θ by

$$xf_\theta=x\theta a \quad\text{and}\quad xg_\theta=a\theta x. \tag{15}$$

Then f_θ and g_θ are transformations of E,

$$f_{\theta\,{}_a\Pi\,\eta}=f_\theta f_\eta \quad\text{and}\quad g_{\theta\,\Pi_a\,\eta}=g_\eta g_\theta. \tag{16}$$

In fact, $xf_{\theta\,{}_a\Pi\,\eta}=x(\theta\Pi_a\eta)a=(x\theta a)\eta a=xf_\theta f_\eta$ for all $x\in E$. This proves the first relation (16); the second one can be similarly obtained. Let h be an arbitrary transformation of E. If θ is defined by

$$x\theta y=xh \quad\text{for all}\quad x, y\in E,$$

then $f_\theta=h$.

Let $\mathscr{T}_E$ denote the full transformation semigroup over E. From the above fact, it is clear that $\theta\to f_\theta$ is a homomorphism of $\mathscr{B}_E({}_a\Pi)$ onto $\mathscr{T}_E$. Likewise $\theta\to g_\theta$ is an anti-homomorphism of $\mathscr{B}_E(\Pi_a)$ onto $\mathscr{T}_E$.

Let ϱ be the congruence on $\mathscr{B}_E({}_a\Pi)$ induced by the homomorphism $\theta\to f_\theta$. In addition we define a relation σ on $\mathscr{B}_E({}_a\Pi)$ as follows:

$\theta\sigma\eta$ for $\theta, \eta\in\mathscr{B}_E$ if and only if $x\theta y=x\eta y$ for all $y\neq a$ and all x. Clearly σ is an equivalence on $\mathscr{B}_E$. Since $|E|>1$, we have $\varrho\neq\omega$, $\sigma\neq\omega$.

To prove $\varrho\cdot\sigma=\omega$, let $\theta, \eta\in\mathscr{B}_E$. We define ξ as follows:

$$x\xi y=x\theta y \ \text{ if } \ y=a, \quad\text{and}\quad x\xi y=x\eta y \ \text{ if } \ y\neq a.$$

Then $\theta\varrho\xi$ and $\xi\sigma\eta$, hence we have proved $\varrho\cdot\sigma=\omega$. By the definition of ϱ and σ, $\theta\varrho\eta$ and $\theta\sigma\eta$ imply $\theta=\eta$, that is, $\varrho\cap\sigma=\iota$. We easily see that

$$\theta\sigma\eta \quad\text{implies}\quad (\zeta{}_a\Pi\theta)\sigma(\zeta{}_a\Pi\eta) \quad\text{for all}\quad \zeta\in\mathscr{B}_E,$$

that is, σ is a left congruence. By Theorem 9, $\mathscr{B}_E(a^*)$ is isomorphic with a left general product of $\mathscr{B}_{E/\sigma}$ by $\mathscr{B}_{E/\varrho}$. For the further study of its structure, we will explain a general case as follows:

Let T be a semigroup, F be a set, $|F|=m$; let x denote a mapping of F into T: $\lambda x=\alpha_\lambda$, $\lambda\in F$, $\alpha_\lambda\in T$. The set of all mappings x of F into T is denoted by S. We define a scalar product $\beta\cdot x$ of $\beta\in T$ and $x\in S$ as follows: if $\lambda x=\alpha_\lambda$ then $\lambda(\beta\cdot x)=\beta\alpha_\lambda$. Clearly $(\beta\gamma)\cdot x=\beta\cdot(\gamma\cdot x)$. Then we define a binary operation on $G=S\times T$ as follows:

$$(x, \alpha)(y, \beta)=(\alpha\cdot y, \alpha\beta). \tag{17}$$

G is a left general product of S by T in which $x\theta_{\alpha,\beta}y=\alpha\cdot y$. It is easy to see that G is a semigroup. The semigroup G with (17) is determined by $m=|F|$ and the semigroup T.

Definition. The semigroup G defined above is denoted by $G=\mathcal{SD}_m(T)$.

Returning to $\mathcal{B}_E({}_a\Pi)$, as we mentioned, the homomorphism $\theta\to f_\theta$ is from $\mathcal{B}_E({}_a\Pi)$ onto $\mathcal{T}_E$. Further each σ-class is associated with a mapping of $E-\{a\}$ into $\mathcal{T}_E$. Accordingly we have the following theorem:

Theorem 10. *$\mathcal{B}_E({}_a\Pi)$ is isomorphic with $\mathcal{SD}_m(\mathcal{T}_E)$ where $m=|E|-1$, $\mathcal{T}_E$ is the full transformation semigroup over E.*

§ 7. Sub-general product

Let U be a subset of $S\overline{\times}T$. We define a notation

$$p_{rj_T}(U)=\{\alpha\in T;\ (x,\alpha)\in U\}.$$

Definition. If U is a subgroupoid of $S\overline{\times}_\Theta T$ and if $p_{rj_T}(U)=T$, then U is called a *sub-general product* of $S\overline{\times}_\Theta T$.

Clearly U is homomorphic onto T, and if $S\overline{\times}_\Theta T$ is a semigroup, U is a subsemigroup.

As is well known, a subdirect product U of groupoids A and T is defined to be a subgroupoid U of the direct product $A\times T$ and $p_{rj_T}(U)=T$ and $p_{rj_A}(U)=A$

In this section we will prove that if a semigroup D is homomorphic onto a semigroup T, then D is isomorphic onto a sub-general product of $S\overline{\times}_\Theta T$ for some set S and some Θ, in other words, any homomorphism φ of D onto T can be extended to a h-homomorphism φ' of some semigroup D' onto T in the sense that $D\subseteqq D'$ and $\varphi'(x)=\varphi(x)$ if $x\in D$.

Proposition 11. *Let g be a homomorphism of a semigroup D onto a semigroup T. Then D is restrictedly isomorphic onto a subdirect product of D and T with respect to g and the projection of $D\times T$ onto T.*

Proof. Let $D'=\{(x,xg);\ x\in D\}$. D' is a subsemigroup of the direct product $D\times T$. We define $h:D\to D'$ by $xh=(x,xg)$. It is easy to see that h is an isomorphism of D onto D'. Let p be the projection $D\times T\to T:(x,y)p=y$. Then $g=h\cdot p$. Therefore D is restrictedly isomorphic into $D\times T$ (i.e. onto D') with respect to g and p.

Proposition 11 shows that the existence of a sub-general product $S\overline{\times}_\Theta T$ into which D can be restrictedly embedded. However the concept "direct product" has been used instead of "general product" and D has been chosen as S. Here is a question raised:

Can we choose $|S|$ as small as possible, $|S|\leqq|D|$? Theorems 12 and 13 will answer this question.

Definition (Petrich [7]). A non-empty subset F of a semigroup D is called a *face* of D if the complement of F is an ideal of D.

Definition. Let g be a homomorphism of a semigroup D onto a semigroup T. An element α of T is called *lowly divisible* if α has a divisor β in T (i.e. $\alpha=\beta\gamma$ or $\gamma\beta$ or $\gamma\beta\delta$ for some $\gamma,\delta\in T$ with $|D_\beta|<|D_\alpha|$, $D_\beta g=\beta$, $D_\alpha g=\alpha$).

Theorem 12. *Suppose that a semigroup D is properly homomorphic, but not h-homomorphic onto a semigroup T. Let g be the homomorphism $D\to T$ and let $D=\bigcup_{\alpha\in T} D_\alpha$ be the decomposition of D induced by $g: D_\alpha g=\alpha$. Then there is a semigroup $\bar D$ which satisfies the following conditions:*

(a) *$\bar D$ is restrictedly isomorphic onto some $S_0\overline{\times}T$ with respect to g and the projection $S_0\overline{\times}T\to T$.*
(b) *D is a face of $\bar D$.*
(c) *Let $n=\text{l.u.b.}\{|D_\alpha|;\ \alpha\in T\}$[1]. Define the cardinal number m by $m=n+1$ if n is finite and if there is a lowly divisible element α of T such that $|D_\alpha|=n$, and by $m=n$ otherwise. Then $|S_0|=m$.*

Proof. Since g is prover, $|T|>1$ and $m>1$. For each $\alpha\in T$ let D_α be a set obtained by adjoining new elements to D_α:

$$\bar D_\alpha=D_\alpha\cup G_\alpha,\quad D_\alpha\cap G_\alpha=\emptyset\quad\text{and}\quad G_\alpha\cap G_\beta=\emptyset\qquad(\alpha\neq\beta)$$

such that $|\bar D_\alpha|=m$ for all $\alpha\in T$. In detail we arrange $\{G_\alpha\}$ as follows: $G_\alpha=\emptyset$ if and only if n is finite, $|D_\alpha|=n$, and there is no lowly divisible element α in T with $|D_\alpha|=n$; if n is infinite and $n=|D_\alpha|$, then $|G_\alpha|=1$. If $G_\alpha\neq\emptyset$ we let G_α contain a special element 0_α. Since g is not a h-homomorphism, $\bigcup_{\alpha\in T} G_\alpha\neq\emptyset$.

Now let $\bar D=\bigcup_{\alpha\in T}\bar D_\alpha$. We define a binary operation $(\circ)$ on $\bar D$ as follows:

If $a\in\bar D_\alpha$ and $b\in\bar D_\beta$, and xy denotes the product of elements x and y in D, then set

$$a\circ b=ab\quad\text{if}\quad a\in D_\alpha\quad\text{and}\quad b\in D_\beta,\quad\text{and}\quad a\circ b=0_{\alpha\beta}\quad\text{otherwise.}$$

It is easy to check that $\bar D$ is associative and $a\to\alpha$, $a\in\bar D_\alpha$, is a proper h-homomorphism of $\bar D$ onto T, since $|\bar D_\alpha|$ is constant $m>1$ and $|T|>1$.

By Theorem 6, $\bar D$ is isomorphic onto $S_0\overline{\times}_\Theta T$ for some set $|S_0|=m$. Let S_0 be a set with $|S_0|=m$ and 0 be a special element of S_0. Let f_α be a one-to-one mapping of S_0 onto S_α such that $0f_\alpha=0_\alpha$. Then $\Theta=\{\theta_{\alpha,\beta};\ \alpha,\beta\in T\}$ is given as follows:

$$x\theta_{\alpha,\beta}y=\begin{cases}((xf_\alpha)(yf_\beta))f_{\alpha\beta}^{-1} & \text{if}\quad xf_\alpha\in D_\alpha,\ yf_\beta\in D_\beta,\\ 0 & \text{otherwise}\end{cases}$$

[1] We assume the well-ordering principle. Since $|D_\alpha|\leqq|D|$, the least upper bound exists.

D is isomorphic into $S_0 \overline{\times}_\theta T$ under a mapping $xf_\alpha \to (x, \alpha)$, $\alpha \in T$. It is easy to see that this is a restricted isomorphism with respect to g and the projection $S_0 \overline{\times} T \to T$. Clearly $\bigcup_{\alpha \in T} G_\alpha$ is the complement of D and is an ideal of $\bar{D}$. Therefore D is a face of D. Thus the proof of the theorem has been completed.

Theorem 13. *Suppose that a semigroup D is properly homomorphic, but not h-homomorphic onto a semigroup T. Let g be the homomorphism $D \to T$ and let $D = \bigcup_{\alpha \in T} D_\alpha$ be the decomposition of D induced by g. Then there is a semigroup $\bar{D}$ which contains D such that*

(α) *$\bar{D}$ is restrictedly isomorphic onto some $S_0 \overline{\times} T$ with respect to g and the projection $S_0 \overline{\times} T \to T$;*
(β) *$\bar{D}$ is an inflation of D* (cf. [2]);
(γ) *$|S_0| = n$ where $n =$* l.u.b. $\{|D_\alpha|;\ \alpha \in T\}$.

Proof. Let $\bar{D}_\alpha = D_\alpha \cup G_\alpha$, $|\bar{D}_\alpha| = n$, for each α where $D_\alpha \cap G_\alpha = \emptyset$ and G_α may be empty; $G_\alpha \cap G_\beta = \emptyset$ $(\alpha \neq \beta)$. Choose exactly one element from each D_α: $\{p_\alpha;\ \alpha \in T\}$, $p_\alpha \in D_\alpha$. Let $\bar{D} = \bigcup_{\alpha \in T} \bar{D}_\alpha$. We define an operation $(\circ)$ on $\bar{D}$ as follows:

If $a \in \bar{D}_\alpha$, $b \in \bar{D}_\beta$, then let

$$a \circ b = \begin{cases} ab & \text{if } a \in D_\alpha,\ b \in D_\beta, \\ ap_\beta & \text{if } a \in D_\cdot,\ b \in G_\beta, \\ p_\alpha b & \text{if } a \in G_\alpha,\ b \in D_\beta, \\ p_\alpha p_\beta & \text{if } a \in G_\alpha,\ b \in G_\beta, \end{cases}$$

where the products on the right side are in D.

It is easy to prove that $\bar{D}$ is a semigroup and an inflation of D and satisfies (α) through (γ). By the way $\theta_{\alpha,\beta}$ is given as follows: Let $x, y \in S_0$, $|S_0| = \bar{D}_\alpha$, and let $f_\alpha\colon S_0 \to \bar{D}_\alpha$ be a one-to-one, onto mapping.

$$x\theta_{\alpha,\beta}y = \begin{cases} ((xf_\alpha)(yf_\beta))f_{\alpha\beta}^{-1} & \text{if } xf_\alpha \in D_\alpha,\ yf_\beta \in D_\beta, \\ ((xf_\alpha)p_\beta)f_{\alpha\beta}^{-1} & \text{if } xf_\alpha \in D_\alpha,\ yf_\beta \in G_\beta, \\ (p_\alpha(yf_\beta))f_{\alpha\beta}^{-1} & \text{if } xf_\alpha \in G_\alpha,\ yf_\beta \in D_\beta, \\ (p_\alpha p_\beta)f_{\alpha\beta}^{-1} & \text{if } xf_\alpha \in G_\alpha,\ yf_\beta \in G_\beta. \end{cases}$$

Remark. $|S_0|$ in Theorem 12 is not necessarily minimum of $|S|$ for which $\bar{D}$ can be embedded into $S \overline{\times} T$ in our sense, strictly speaking, $|S_0|$ is either minimum or minimum plus one, while $|S_0|$ in Theorem 13 is certainly minimum. In Theorem 12, even if D is s-indecomposable, that is, if D has no proper semilattice homomorphic image, then $S_0 \overline{\times} T$ is not. In Theorem 13, however, if D is s-indecomposable, $S_0 \overline{\times} T$ is also; but $S_0 \overline{\times} T$ is not simple even if D is simple. On the other hand in Proposition 11, if D is simple, $D \times T$ is simple.

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Subalgebra and automorphism structure in universal algebras; a concrete characterization

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§ 1. Introduction

E. T. SCHMIDT [4] has shown that, from an abstract viewpoint, the automorphism structure and subalgebra structure of algebras are completely independent of each other (Theorem 1). The main purpose of this note is to give a concrete version of Schmidt's result where isomorphism is replaced by equality (Theorem 4). We also show how Schmidt's theorem may be derived from the corresponding concrete result.

By an *algebra* $\mathfrak{A}=\langle A, F\rangle$, we mean a *universal algebra* with base set A and a (possibly infinite) set of finitary operations F. We denote by $\operatorname{Aut}\mathfrak{A}$ the automorphism group of $\mathfrak{A}$ and by $\operatorname{Su}\mathfrak{A}$ the closure structure consisting of subsets of A which are subalgebras of $\mathfrak{A}$. For $B\subseteqq A$ we use $\mathbf{C}B$ to denote the subalgebra of $\mathfrak{A}$ generated by B, thus $\mathbf{C}B\in\operatorname{Su}\mathfrak{A}$. $\operatorname{Su}\mathfrak{A}$ is always an algebraic closure structure; when viewed as a partially ordered set $\mathfrak{L}=\langle\operatorname{Su}\mathfrak{A}, \subseteqq\rangle$ is referred to as the *subalgebra lattice* of $\mathfrak{A}$, and this lattice is always compactly generated. Schmidt's result can be stated in the following form:

Theorem 1. [4] *Given any group* $\mathfrak{G}$ *and any algebraic (compactly generated) lattice* $\mathfrak{L}$, *there is an algebra* $\mathfrak{A}$ *with* $\mathfrak{G}$ *isomorphic to* $\operatorname{Aut}\mathfrak{A}$ *and* $\mathfrak{L}$ *isomorphic to* $\langle\operatorname{Su}\mathfrak{A}, \subseteqq\rangle$.

§ 2. Concrete characterization

To characterize $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Su}\mathfrak{A}$ *concretely* one must specify for a given set A which subgroups G of the permutation group on A are compatible with various algebraic closure structures L defined on subsets of A, in the sense that $G=\operatorname{Aut}\mathfrak{A}$ and $L=\operatorname{Su}\mathfrak{A}$ for suitable $\mathfrak{A}=\langle A, F\rangle$. Separate concrete characterization theorems for $\operatorname{Aut}\mathfrak{A}$ and $\operatorname{Su}\mathfrak{A}$ have been given by other authors. B. JONSSON has characterized the groups of permutations of A which are equal to $\operatorname{Aut}\mathfrak{A}$ for some algebra $\mathfrak{A}=\langle A, F\rangle$, viz.

Theorem 2. [3] *For a subgroup G of the permutation group on a set A to be equal to* $\mathrm{Aut}\,\mathfrak{A}$ *for some algebra with base set A it is necessary and sufficient that G be locally closed in the following sense: if φ is a permutation of A and on each finite subset of A, φ agrees with some $\psi \in G$, then $\varphi \in G$ as well.*

The concrete characterization of $\mathrm{Su}\,\mathfrak{A}$ is due to G. BIRKHOFF and O. FRINK:

Theorem 3. [1] *If L is any algebraic closure structure consisting of subsets of A then $L = \mathrm{Su}\,\mathfrak{A}$ for some algebra $\mathfrak{A} = \langle A, F \rangle$.*

For the combined concrete characterization the most general situation one could expect (complete independence) would be that every locally closed group should be compatible (in the above sense) with every algebraic closure structure definable on A. There is however some interplay between $\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Su}\,\mathfrak{A}$, as the following theorem shows.

Theorem 4. *If G is a group of permutations on A and L is an algebraic closure structure consisting of subsets of A, then there is an algebra $\mathfrak{A} = \langle A, F \rangle$ such that $G = \mathrm{Aut}\,\mathfrak{A}$ and $L = \mathrm{Su}\,\mathfrak{A}$ iff the following conditions are satisfied:*

(1) *G is locally closed;*
(2) $B \in L$, $\varphi \in G \Rightarrow \varphi(B) \in L$;
(3) *for each finite subset $X \subseteq A$, if σ and $\tau \in G$ agree on X, then they agree on $\mathrm{C}X$ as well.*

§ 3. Proof of Theorem 4

The necessity of (2) and (3) is obvious, and (1) of course follows from Theorem 2. To prove the converse of the theorem, assume (1), (2) and (3) hold for some A and suitable G and L. We construct an algebra $\mathfrak{A}$ with base set A as follows: There will be two kinds of operations, $f_{(y,a)}$ and F_z, indexed by sets I and J respectively where

$$I = \{(y, a) \mid y \text{ is a finite 1—1 sequence of elements of } A \text{ and } a \in \mathrm{C}X - X,\ X = \mathrm{Range}\, y\}$$

and

$$J = \{z \mid z \text{ is a finite 1—1 sequence of elements of } A\}.$$

The operations are defined in the following way: For $(y, a) \in I$ with the length of $y = n > 0$, $f_{(y,a)}$ is n-ary and

$$f_{(y,a)}(w_0 \dots w_{n-1}) = \begin{cases} \sigma a & \text{if } \ w = \sigma y \ \text{ for some } \ \sigma \in G, \\ w_0 & \text{otherwise.} \end{cases}$$

For $(y, a) \in I$ with the length of $y = 0$, $f_{(y,a)}$ is nullary and $f_{(y,a)} = a$. For $z \in J$ of length

n, F_z is n-ary and

$$F_z(w_0 \dots w_{n-1}) = \begin{cases} w_0 & \text{if} \quad w = \sigma y \quad \text{for some} \quad \sigma \in G, \\ w_1 & \text{otherwise.} \end{cases}$$

Note that $f_{(y,a)}$ is well defined since (3) yields $\sigma y = \tau y \Rightarrow \sigma a = \tau a$.

Now let $\mathfrak{A} = \langle A; \langle f_i | i \in I \rangle \cup \langle F_j | j \in J \rangle \rangle$. It will suffice to show that $G = \operatorname{Aut} \mathfrak{A}$ and $L = \operatorname{Su} \mathfrak{A}$.

To see that $L = \operatorname{Su} \mathfrak{A}$, let $B \in \operatorname{Su} \mathfrak{A}$. Note $B \in L$ iff $\mathbf{C}B = B$. Now if $B \notin L$ one has $\mathbf{C}B \neq B$ so there is some $a \in \mathbf{C}B \sim B$. Since L is algebraic there is some finite $X \subseteqq B$ with $\mathbf{C}X = \mathbf{C}B$ and $a \in \mathbf{C}X - X$. If $X = \emptyset$ then $f_{(\emptyset, a)} = a \in B$, so $X \neq \emptyset$. Let y be a $1-1$ sequence whose range is X; then $f_{(y,a)}(y) = a \in B$, contradicting the choice of a. Thus $\operatorname{Su} \mathfrak{A} \subseteqq L$. To establish the opposite inclusion it is only necessary to select $B \in L$ and verify that B is closed under the operations of $\mathfrak{A}$. This is obviously true for operations F_z, $z \in J$. Now let $f_{(y,a)}$ be an operation of $\mathfrak{A}$ of rank n. If $n = 0$, i.e. y = empty sequence, then $a \in \mathbf{C}\emptyset \subseteqq \mathbf{C}B = B$ thus $f_{(y,a)} = a \in B$. If $n > 0$ say $y = \langle y_0 \dots y_{n-1} \rangle$, let X = range y and let $b = \langle b_0, \dots, b_{n-1} \rangle \in B^n$. If $\sigma y \neq b$ for any $\sigma \in G$ then $f_{(y,a)}(b) = b_0 \in B$. If $\sigma y = b$ for some $\sigma \in G$ then $f_{(y,a)}(b) = \sigma a$; but $\sigma(X) \subseteqq B$ so $X \subseteqq \sigma^{-1}(B)$ and $\mathbf{C}X \subseteqq \mathbf{C}\sigma^{-1}B$ thus $a \in \mathbf{C}\sigma^{-1}B$. Further since $B \in L$ we have $\sigma^{-1}B \in L$ so $\mathbf{C}\sigma^{-1}B = \sigma^{-1}B$ and it follows that $\sigma a \in B$ as desired. This shows that $L = \operatorname{Su} \mathfrak{A}$.

It remains only to see that $G = \operatorname{Aut} \mathfrak{A}$. Let $\varphi \in G$. To verify that $\varphi \in \operatorname{Aut} \mathfrak{A}$ one must check that φ is substitutive over each of the operations F_z, and $f_{(y,a)}$. By noting that $y = \sigma z$ iff $\varphi y = \varphi \sigma z$ for $\sigma \in G$, it is clear that $\varphi F_z(y) = F_z(\varphi y)$. The check for $f_{(y,a)}$ is straightforward, and establishes $G \subseteqq \operatorname{Aut} \mathfrak{A}$. Now let φ be a permutation of A, $\varphi \notin G$. We will show $\varphi \notin \operatorname{Aut} \mathfrak{A}$, and it follows that $G = \operatorname{Aut} \mathfrak{A}$. G is locally closed therefore there is a finite set $D \subseteqq A$ such that no member of G agrees with φ on D. Let z be a 1—1 sequence with range D. Then F_z is an $\mathfrak{A}$ operation, and $\varphi F_z(z) = \varphi z_0$, whereas $F_z(\varphi z) = \varphi z_1$. Since z is 1—1 and φ is a permutation, $\varphi F_z(z) \neq F_z(\varphi z)$, thus $\varphi \notin \operatorname{Aut} \mathfrak{A}$. This completes the proof of Theorem 4.

§ 4. Theorem 1 as a corollary

Given a group $\mathfrak{G} = \langle G, \cdot \rangle$ and an algebraic lattice $\mathfrak{L}$, we exhibit a set A, a group of permutations of A, $\mathfrak{G}^+ \cong \mathfrak{G}$, and an algebraic closure structure, L^+, on subsets of A, with $\langle L^+, \subseteqq \rangle \cong \mathfrak{L}$ (lattice isomorphism). Further $\mathfrak{G}^+$ and L^+ will satisfy (1)(2)(3) of Theorem 4 so that one may conclude $\mathfrak{G} \cong \operatorname{Aut} \mathfrak{A}$ and $\mathfrak{L} \cong \langle \operatorname{Su} \mathfrak{A}, \subseteqq \rangle$ for some algebra $\mathfrak{A}$. The abstract theorem in a way asserts the existence of a concrete object; the only problem is first to find a suitable set for the application of Theorem 4. As a first step note that by well-known lattice theoretic considerations, we may assume that (within isomorphism) $\mathfrak{L}$ is a lattice of subsets of some set B, and furthermore that the zero element of $\mathfrak{L}$ is the null set. Now let $A=\{(x, y)|x\in B, y\in G\}$. For each $g\in G$ define $g^+: A\to A$ by $g^+(x, y) = (x, g\cdot y)$. Let $G^+=\{g^+|g\in G\}$, and $\mathfrak{G}^+=\langle G^+, 0\rangle$; clearly $\varphi:\mathfrak{G}\to\mathfrak{G}^+$ by $\varphi(g)=g^+$ is a group isomorphism. Note that $\mathfrak{G}\cong\mathrm{Aut}\,\langle G; f_g\rangle_{g\in G}$ where f_g is the unary operation "right multiplication by g". Since $\mathrm{Aut}\,\langle G; f_g\rangle_{g\in G}$ is locally closed (by Theorem 2) it follows that $\mathfrak{G}^+$ is locally closed in A^A. Now for each $P\in L$ let $P^+=\{(x, y)\in A|x\in P, y\in G\}$, and set $L^+=\{P^+|P\in L\}$. L^+ is an algebraic closure structure composed of subsets of A, and one easily verifies $\langle L^+, \subseteq\rangle \cong \mathfrak{L}$ (lattice isomorphism). It is clear from the method of construction that $\mathfrak{G}^+$ and L^+ satisfy (1) and (2) of theorem 4. To see that (3) holds as well, let $X\subseteq A$. If $X=\emptyset$ then (3) holds vacuously since $\emptyset\in L^+$. If X is finite and non-empty, say $(x_0, y_0)\in X$, suppose $g^+, h^+\in G^+$ agree on X. Then $y_0\cdot g = y_0\cdot h$, but $y_0\in G$ so $g=h$ and thus $g^+=h^+$ so of course (3) is satisfied.

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A theorem on factorizable groups

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To Professor Ladislaus Rédei on his seventieth birthday

The purpose of this note is to prove the following theorem.

Theorem. *Let a finite group $\mathfrak{G}$ be the product of two subgroups $\mathfrak{A}$ and $\mathfrak{B}$ such that* (1) *$\mathfrak{A}$ and $\mathfrak{B}$ have non-trivial centers*; (2) *if B is a non-identity element of $\mathfrak{B}$, then the centralizer of B in $\mathfrak{G}$ is contained in $\mathfrak{B}$. Then $\mathfrak{G}$ is not simple.*

Remark. Take $\mathfrak{G}$, $\mathfrak{A}$ and $\mathfrak{B}$ as the icosahedral group, a Sylow 5-subgroup and a tetrahedral subgroup respectively. Then all the conditions in the theorem except (1) to $\mathfrak{B}$ are satisfied. This shows that (1) applied to $\mathfrak{A}$ and (2) are not sufficient to imply the non-simplicity of $\mathfrak{G}$.

Notation. Let $\mathfrak{X}$ be a finite group. $Z(\mathfrak{X})$ denotes the center of $\mathfrak{X}$. For a prime p, $\mathfrak{X}_p$ denotes a Sylow p-subgroup of $\mathfrak{X}$. Let $\mathfrak{X}$ be a subset of $\mathfrak{X}$. $|\mathfrak{Y}|$ denotes the number of elements in $\mathfrak{Y}$. Ns $\mathfrak{Y}$ denotes the normalizer of $\mathfrak{Y}$ in $\mathfrak{X}$. Cs $\mathfrak{Y}$ denotes the centralizer of $\mathfrak{Y}$ in $\mathfrak{X}$. If $\mathfrak{Y}=\{Y\}$, Cs $Y=$Cs $\mathfrak{Y}$. For $X\in\mathfrak{X}$, $[X]$ denotes the conjugacy class of $\mathfrak{X}$ containing X. $\mathfrak{E}=\{E\}$ denotes the identity subgroup of $\mathfrak{X}$. Let $\mathscr{X}$ be a set of irreducible characters of $\mathfrak{X}$. Then $\Gamma(\mathscr{X})$ denotes the ring of rational integral linear linear combinations λ of characters in $\mathscr{X}$. $\Gamma_0(\mathscr{X})$ denotes the subring of $\Gamma(\mathscr{X})$ consisting of all λ with $\lambda(E)=0$. Let φ be a character of a subgroup $\mathfrak{Y}$. Then φ^* denotes the character of $\mathfrak{X}$ induced by φ. $1_{\mathfrak{X}}$ denotes the principal character of $\mathfrak{X}$.

Proof. Assume that the theorem is false. Let $\mathfrak{G}$ be a simple group satisfying all the conditions in the theorem.

(i) By a theorem of BURNSIDE ([3], p. 491) $|\mathfrak{B}|$ is not a prime power.

(ii) $\mathfrak{A}\cap\mathfrak{B}=\mathfrak{E}$. In fact, otherwise take $B(\neq E)\in\mathfrak{A}\cap\mathfrak{B}$. Then $Z(\mathfrak{A})\leqq$Cs $B\leqq\mathfrak{B}$. This shows that $\mathfrak{B}$ contains a normal subgroup of $\mathfrak{G}$ containing $Z(\mathfrak{A})$. This is a contradiction.

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(iii) $\mathfrak{A}$ and $\mathfrak{B}$ are Hall subgroups of $\mathfrak{G}$. In fact, otherwise, let p be a common prime divisor of $|A|$ and $|\mathfrak{B}|$. We may assume that $\mathfrak{G}_p=\mathfrak{A}_p\mathfrak{B}_p$ ([3], p. 676). Then $\mathfrak{A}_p\neq\mathfrak{E}\neq\mathfrak{B}_p$. Take $P(\neq E)\in\mathfrak{B}_p$. Then $Z(\mathfrak{G}_p)\subseteqq \mathrm{Cs}\, P\subseteqq\mathfrak{B}$. Take $P^*(\neq E)\in Z(\mathfrak{G}_p)$. Then $\mathfrak{A}_p\subseteqq\mathfrak{G}_p\subseteqq \mathrm{Cs}\, P^*\subseteqq\mathfrak{B}$. This contradicts (ii).

(iv) $\mathfrak{B}$ is a T.I. set. Namely if $X^{-1}\mathfrak{B}X\neq\mathfrak{B}$ for $X\in\mathfrak{G}$, then $X^{-1}\mathfrak{B}X\cap\mathfrak{B}=\mathfrak{E}$. In fact, otherwise, take $B(\neq E)\in X^{-1}\mathfrak{B}X\cap\mathfrak{B}$. Then $Z(X^{-1}\mathfrak{B}X)\subseteqq \mathrm{Cs}\, B\subseteqq\mathfrak{B}$. So $X^{-1}\mathfrak{B}X\subseteqq \mathrm{Cs}\, Z(X^{-1}\mathfrak{B}X)\subseteqq\mathfrak{B}$. Thus $X^{-1}\mathfrak{B}X=\mathfrak{B}$. This is a contradiction.

(v) $\mathrm{Ns}\,\mathfrak{B}$ is a Frobenius group with $\mathfrak{B}$ the kernel. In fact, if $\mathrm{Ns}\,\mathfrak{B}=\mathfrak{B}$, then by (iv) and by a theorem of FROBENIUS ([3], p. 495) $\mathfrak{G}$ is not simple. Thus $\mathrm{Ns}\,\mathfrak{B}\neq\mathfrak{B}$. Let $\mathfrak{C}$ be a complement of $\mathfrak{B}$ in $\mathrm{Ns}\,\mathfrak{B}$ ([3], p. 126). Then by the condition (2) on $\mathfrak{B}$ and by (iii) no element $(\neq E)$ of $\mathfrak{C}$ commutes with an element $(\neq E)$ of $\mathfrak{B}$. Hence we get our assertion ([3], p. 497).

By a theorem of THOMPSON ([3], p. 499) $\mathfrak{B}$ is nilpotent.

(vi) By (iv) and (v) we are in a position to apply the theory of exceptional characters. Let $\mathscr{S}$ be the set of irreducible characters of $\mathrm{Ns}\,\mathfrak{B}$ which do not have $\mathfrak{B}$ in their kernel. Then the induction map * is a linear isometry from $\Gamma_0(\mathscr{S})$ into the character ring of $\mathfrak{G}$ such that $\lambda^*(E)=0$ for $\lambda\in\Gamma_0(\mathscr{S})$ ([2], (23. 1), (25. 4)). If $|\mathfrak{C}|+1 = |\mathfrak{B}|$, then $|\mathfrak{B}|$ must be a prime power. This contradicts (i). Hence ($\mathfrak{C}$, *) is coherent ([2], (31. 6)). Namely * can be extended to a linear isometry $^{\mathscr{C}}$ from $\Gamma(\mathscr{S})$ into the character ring of $\mathfrak{G}$.

(vii) Let χ be an irreducible component of $l_{\mathfrak{A}}^*$. Assume that $\chi\neq l_{\mathfrak{G}}$ and that $\pm\chi\notin\mathscr{S}^{\mathscr{C}}$. Then there exists a rational integer c such that $\chi(B)=c$ for every $B(\neq E)\in\mathfrak{B}$. We show that

$$\chi(E) = |\mathfrak{B}|-1.$$

In fact, since

$$\sum_{B\in\mathfrak{B}}\chi(B)1_{\mathfrak{B}}(B) = \chi(E)+c(|\mathfrak{B}|-1) = m|\mathfrak{B}|,$$

where m is a non-negative integer, we obtain that

$$\chi(E)-c = (m-c)|\mathfrak{B}|.$$

Since $\chi(E)\leqq|\mathfrak{B}|-1$ and since $\mathfrak{G}$ is simple, we see that

$$\chi(E)>|c|,\ 0>c,\ m-c = 1,\ m=0 \quad\text{and}\quad c=-1.$$

Thus $\chi(E) = |\mathfrak{B}|-1$. This implies that the permutation representation of $\mathfrak{G}$ induced by $\mathfrak{A}$ is doubly transitive ([2], (9. 6)). Here we notice that $l_{\mathfrak{A}}^*$ is the character of this permutation representation.

Now let A be an element of $Z(\mathfrak{A})$ of order p, a prime. Then since $l_{\mathfrak{A}}^*$ is doubly transitive, $l_{\mathfrak{A}}^*(A)=1$. If $X^{-1}AX\in\mathfrak{A}$ for $X\in\mathfrak{G}$, then X fixes the "point" $\mathfrak{A}$. Hence $X\in\mathfrak{A}$ and $X^{-1}AX=A$.

Suppose that there exists a q-subgroup $\mathfrak{Q}$ of $\mathfrak{G}$ such that $q \neq p$ and A induces a non-trivial automorphism of $\mathfrak{Q}$. If q divides $|\mathfrak{B}|$, then we may assume that $\mathfrak{Q} \subseteqq \mathfrak{B}$: A normalizes $\mathrm{Cs}\,\mathfrak{Q}$. Let $\mathfrak{R}$ be the Sylow q-complement of $\mathfrak{B}$. Then since $\mathfrak{B} \supseteqq$ $\supseteqq \mathrm{Cs}\,\mathfrak{Q} \supseteqq \mathfrak{R}$, A normalizes $\mathfrak{R}$ and hence $\mathfrak{B}$. Then $\mathrm{Ns}\,\mathfrak{B}$ contains a normal subgroup of $\mathfrak{G}$ containing $\mathfrak{A}$. This is a contradiction. Therefore q divides $|\mathfrak{A}|$. A normalizes $\mathrm{Cs}\,\mathfrak{Q}$. Since $\mathrm{Cs}\,\mathfrak{Q}$ contains some conjugate of $Z(\mathfrak{A})$, there exists a Sylow p-subproup $\mathfrak{D}(\neq \mathfrak{E})$ of $\mathrm{Cs}\,\mathfrak{Q}$ such that A normalizes $\mathfrak{D}$. Obviously $A \notin \mathfrak{D}$. $\mathfrak{D}$ contains some conjugate $Y^{-1}AY$ of A. Thus $A^{-1}Y^{-1}AY \neq E$ is a p-element. Since $A \in Z(\mathfrak{A})$, we have that

$$[A^{-1}][A] = |\mathfrak{B}|[E] + r[A^{-1}Y^{-1}AY],$$

where r is a positive integer.[1]) Since $\chi(A)=0$, this implies that $\chi(A^{-1}Y^{-1}AY)$ is a negative integer. Hence $l_{\mathfrak{A}}^{*}(A^{-1}Y^{-1}AY) \leqq 0$. Since $A^{-1}Y^{-1}AY$ is a p-element, this is a contradiction. Therefore there exists no subgroup such as $\mathfrak{Q}$. Hence by a theorem of SHULT ([4]) $\mathfrak{G}$ is not simple. Thus for every irreducible component $\zeta \neq l_{\mathfrak{G}}$ of $l_{\mathfrak{A}}^{*}$ we must have $\pm\zeta \in \mathcal{S}^{\mathcal{C}}$.

(viii) Now we can follow an arguement due to Burnside as follows ([1], § 151). Put $\mathcal{S} = \{\zeta_i,\ 1 \leqq i \leqq s\}$ with $s = |\mathcal{S}|$. If $s=2$, then $|\mathfrak{B}|$ is a prime power against (i). Hence $s \geqq 3$. By ([2], (23. 1)) and by the coherence of $(\mathcal{S}, {}^{*})$ we have the following equation:

(a) $$(e_j\zeta_i - e_i\zeta_j)^{*} = e_j\zeta_i^{*} - e_i\zeta_j^{*} = e_j\zeta_i^{\mathcal{C}} - e_i\zeta_j^{\mathcal{C}},$$

where $1 \leqq i, j \leqq s$ and $e_k = \zeta_k(E)/|\mathfrak{C}|$ for $k=1, \ldots, s$. (a) implies that

$$\sum_{X \in \mathfrak{G}} 1_{\mathfrak{A}}^{*}(X)\big(e_j\zeta_i^{\mathcal{C}}(X) - e_i\zeta_j^{\mathcal{C}}(X)\big) = 0.$$

Therefore the decomposition of $l_{\mathfrak{A}}^{*}$ into its irreducible components has the following form:

(b) $$1_{\mathfrak{A}}^{*} = 1_{\mathfrak{G}} + m\sum_{i=1}^{s} e_i\zeta_i^{\mathcal{C}},$$

where m is a rational integer. By (b) we see that $\zeta_i^{\mathcal{C}}(E) > 0$ for all i or $\zeta_i^{\mathcal{C}}(E) < 0$ for all i.

We show that $|\zeta_j^{\mathcal{C}}(E)| \geqq \zeta_j(E)$ for all j. By (a) and by the Frobenius reciprocity theorem ([2], (9. 4)), this is obvious, if $\zeta_j^{\mathcal{C}}(E) > 0$ or if $-\zeta_j^{\mathcal{C}}$ appears as an irreducible component of ζ_j^{*}. Hence we may assume that $\zeta_j^{\mathcal{C}}(E) < 0$ and that $-\zeta_j^{\mathcal{C}}$ does not appear as an irreducible component of ζ_j^{*}. Then by (a) we see that $-\zeta_j^{*}$ appears as an irreducible component of ζ_i^{*} with the multiplicity $\zeta_i(E)/\zeta_j(E)$, where $j \neq i$. This implies that $\zeta_i(E) \geqq \zeta_j(E)$ and that $|\zeta_j^{\mathcal{C}}(E)| \geqq \zeta_i(E)$([2], (9. 4)).

[1]) The usefulness of this equation we owe to Professor H. WIELANDT.

Now since $\sum_{i-1}^{s}\zeta_i(E)^2 = |\mathfrak{B}|\,|\mathfrak{C}|-|\mathfrak{C}|$, (b) implies that $|m|=1$ and $|\zeta_i^{\mathfrak{C}}(E)|=\zeta(E_i)$. Then since $s\geqq 3$, we obtain that $\zeta_i^{\mathfrak{C}}(E)>0$ and that $\zeta_i^{\mathfrak{C}}$ restricted to $\mathrm{Ns}\,\mathfrak{B}$ is equal to ζ_i. $(i=1, \dots, s)$.

By a) we obtain that

$$e_j\zeta_i^{\mathfrak{C}}(A) = e_i\zeta_j^{\mathfrak{C}}(A)$$

for every element A of $\mathfrak{G}$ which is not conjugate to some element $(\neq E)$ of $\mathfrak{B}$. Since the number of characters ζ_i with $\zeta_i(E)=|\mathfrak{C}|$ equals $((\mathfrak{B}:\mathfrak{B}')-1)/|\mathfrak{C}|$, where $\mathfrak{B}'$ denotes the commutator subgroup of $\mathfrak{B}$, we may assume that $\zeta_1(E)=\zeta_2(E)=|\mathfrak{C}|$. Then

$$\zeta_i^{\mathfrak{C}}(A) = e_i\zeta_1^{\mathfrak{C}}(A)$$

for all i. Now by (b) be obtain that

(c) $$1_{\mathfrak{A}}^*(A) = 1+t\zeta_1^{\mathfrak{C}}(A)$$

for every element A of $\mathfrak{G}$ which is not conjugate to some element $(\neq E)$ of $\mathfrak{B}$, where $t = (|B|-1)/|\mathfrak{C}|$. Let $\mathfrak{G}(i)$ be the set of elements G in $\mathfrak{G}$ such that $l_{\mathfrak{A}}^*(G)=i$. Since $t\geqq s\geqq 3$, by (c) we see that $\mathfrak{G}(0)$ coincides with the set of elements of $\mathfrak{G}$ which are conjugate to elements $(\neq E)$ of $\mathfrak{B}$. Then we have that

$$\sum_{B\in\mathfrak{G}(0)}\zeta_1^{\mathfrak{C}}(B) = (\mathfrak{A}:\mathfrak{C})\sum_{B\in\mathfrak{B}-\{E\}}\zeta_1(B) = -|\mathfrak{A}|$$

and that

$$\sum_{B\in\mathfrak{G}(0)}\zeta_1^{\mathfrak{C}}(B)\overline{\zeta_2^{\mathfrak{C}}(B)} = (\mathfrak{A}:\mathfrak{C})\sum_{B\in B-\{E\}}\zeta_1(B)\overline{\zeta_2(B)} = -|\mathfrak{A}|\,|\mathfrak{C}|.$$

Finally, since $\sum_{G\in\mathfrak{G}}\zeta_1^{\mathfrak{C}}(G) = \sum_{G\in\mathfrak{G}}\zeta_1^{\mathfrak{C}}(G)\overline{\zeta_2^{\mathfrak{C}}(G)} = 0$, by (c) we obtain that

(d) $$-|\mathfrak{A}|+|\mathfrak{G}(t+1)|+2|\mathfrak{G}(2t+1)|+\dots+|\mathfrak{C}| = 0$$

and

(e) $$-|A|\,|\mathfrak{G}|+|\mathfrak{G}(t+1)|+4|\mathfrak{G}(2t+1)|+\dots+|\mathfrak{C}|^2 = 0.$$

(d) and (e) enforce $\mathfrak{G}(t+1)$, $\mathfrak{G}(2t+1)$, ... to be empty. Hence $|\mathfrak{A}|=|\mathfrak{C}|$ and $\mathfrak{A}=\mathfrak{C}$. This is a contradiction.

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Permutation polynomials in several variables

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1. Introduction. In [11] W. NÖBAUER introduced the notion of a permutation polynomial in several variables over a commutative ring with identity, where the polynomial is considered modulo an ideal. We apply this definition to polynomials in several variables with integral coefficients.

Let Z denote the ring of integers and let p be a fixed prime. For a given $n \geqq 1$, we consider lattice points $(a_1, \dots, a_n)$, $a_i \in Z$, $1 \leqq i \leqq n$. Two lattice points $(a_1, \dots, a_n)$, $(b_1, \dots, b_n)$ are said to be congruent modulo p if $a_i \equiv b_i \pmod p$ for all $i = 1, \dots, n$. By means of this definition, the set of n-dimensional lattice points is divided into exactly p^n equivalence classes. In the sequel, M_p^n will stand for a complete system of representatives from those equivalence classes. We give the following

Definition 1. A polynomial $f \in Z[x_1, \dots, x_n]$ is called a permutation polynomial mod p if the congruence $f(x_1, \dots, x_n) \equiv a \pmod p$ has exactly p^{n-1} solutions in M_p^n for each $a = 0, 1, \dots, p-1$.

Remark. The above definition is obviously independent of the choice of M_p^n. The definition coincides with Nöbauer's definition for permutation polynomials over Z modulo the ideal (p) (see [11], p. 342).

For $n = 1$, the theory of permutation polynomials is well developed ([1]; [3]; [4]; [5]; [7], ch. 18; [8], ch. 5; [10]; [12]; [13]; [14]). We therefore suppose $n \geqq 2$ from now on. Some results for the case $n = 2$ have been obtained by KURBATOV and STARKOV [9]. In this paper, two necessary and sufficient conditions for permutation polynomials mod p are given and all permutation polynomials mod p of degree 1 and degree 2 are characterized. Generalizations to Galois fields shall be discussed elsewhere.

2. Two criteria. First we show the following

Theorem 1. *$f \in Z[x_1, \dots, x_n]$ is a permutation polynomial* mod p *if and only if each congruence $f(x_1, \dots, x_n) \equiv a \pmod p$, $a = 0, 1, \dots, p-1$, has at least one solution and*

$$\sum_{(a_1, \dots, a_n) \in M_p^n} [f(a_1, \dots, a_n)]^{tp^{n-2}} \equiv 0 \pmod{p^{n-1}} \quad \textit{for} \quad t = 1, \dots, p-1.$$

Proof. Put k_a=number of solutions from M_p^n of $f(x_1, \ldots, x_n) \equiv a(\bmod p)$, $a=0, 1, \ldots, p-1$. Since $c \equiv d(\bmod p)$ implies $c^{p^{n-2}} \equiv d^{p^{n-2}}(\bmod p^{n-1})$, we get

$$\sum_{(a_1,\ldots,a_n)\in M_p^n} [f(a_1, \ldots, a_n)]^{tp^{n-2}} \equiv \sum_{a=0}^{p-1} k_a a^{tp^{n-2}} \pmod{p^{n-1}} \quad \text{for} \quad t = 1, \ldots, p-1.$$

Suppose now that f is a permutation polynomial mod p; then $k_a = p^{n-1}$ for all $a=0, 1, \ldots, p-1$ and we are done.

Conversely, suppose that the condition of the theorem is satisfied. Then

$$\sum_{a=0}^{p-1} k_a a^{tp^{n-2}} \equiv 0 \pmod{p^{n-1}} \quad \text{for all} \quad t = 1, \ldots, p-1.$$

Since the above congruence also holds for $t=0$ (with $0^0=1$), we get a system of homogeneous linear equations in $k_0, \ldots, k_{p-1}$ over the residue class ring modulo p^{n-1} with determinant D being a Vandermonde determinant. Thus

$$D = \prod_{0 \leq i < j \leq p-1} (j^{p^{n-2}} - i^{p^{n-2}}).$$

Since $i^{p^{n-2}} \equiv j^{p^{n-2}}(\bmod p)$ would imply $i \equiv j(\bmod p)$, we have $D \not\equiv 0(\bmod p)$, i.e. D is not a zero divisor in the residue class ring modulo p^{n-1}. Therefore $k_a \equiv 0(\bmod p^{n-1})$ for $a=0, 1, \ldots, p-1$. By hypothesis, $k_a \geq 1$ for all $a=0, 1, \ldots, p-1$ and so $k_a \geq p^{n-1}$ for all $a = 0, 1, \ldots, p-1$. From $\sum_{a=0}^{p-1} k_a = p^n$ it follows that $k_a = p^{n-1}$ for all $a = 0, 1, \ldots, p-1$.

Theorem 2. *$f \in Z[x_1, \ldots, x_n]$ is a permutation polynomial* mod p *if and only if*

$$\sum_{(a_1,\ldots,a_n)\in M_p^n} e^{2\pi i \frac{m}{p} f(a_1,\ldots,a_n)} = 0 \quad \textit{for all} \quad m = 1, \ldots, p-1.$$

Proof. Again putting k_a=number of solutions from M_p^n of $f(x_1, \ldots, x_n) \equiv$ $\equiv a(\bmod p)$, $a=0, 1, \ldots, p-1$, we have

$$\sum_{(a_1,\ldots,a_n)\in M_p^n} e^{2\pi i \frac{m}{p} f(a_1,\ldots,a_n)} = \sum_{a=0}^{p-1} k_a e^{2\pi i \frac{m}{p} a} \quad \text{for} \quad m = 1, \ldots, p-1.$$

So if $k_a = p^{n-1}$ for all $a=0, 1, \ldots, p-1$, then the necessity of the condition follows easily.

Conversely, suppose that $\sum_{a=0}^{p-1} k_a e^{2\pi i \frac{m}{p} a} = 0$ for all $m=1, \ldots, p-1$. This gives rise to the following system of linear equations for $k_0, k_1, \ldots, k_{p-1}$:

$$k_0 + k_1 + \cdots + k_{p-1} = p^n,$$

$$\sum_{a=0}^{p-1} k_a e^{2\pi i \frac{m}{p} a} = 0 \qquad (m = 1, \ldots, p-1).$$

The determinant Δ of this system is a Vandermonde determinant, hence

$$\Delta = \prod_{0 \leqq r < s \leqq p-1} \left(e^{2\pi i \frac{s}{p}} - e^{2\pi i \frac{r}{p}}\right) \neq 0.$$

So there is a unique solution to the system, and this solution is $k_0 = k_1 = \dots = k_{p-1} = p^{n-1}$.

Remark. Theorem 2 clearly holds for $n=1$ as well. Actually, Theorem 2 is contained in a general result of CARLITZ [2, Theorem 4. 6.] but we have included the foregoing proof because of its simplicity.

3. Some auxiliary results.

Lemma 1 (NÖBAUER [11]). *If $f \in Z[x_1, \dots, x_n]$ can be written in the form $f(x_1, \dots, x_n) = g(x_1, \dots, x_k) + h(x_{k+1}, \dots, x_n)$, $1 \leqq k < n$, where $h \in Z[x_{k+1}, \dots, x_n]$ is a permutation polynomial* mod p *and $g \in Z[x_1, \dots, x_k]$, then f is a permutation polynomial* mod p.

Lemma 2. *Let $f \in Z[x_1, \dots, x_n]$ be a permutation polynomial* mod p. *If $x_i = a_{i1}y_1 + a_{i2}y_2 + \dots + a_{in}y_n + b_i$ ($a_{ij} \in Z$, $b_i \in Z$, $1 \leqq i \leqq n$, $1 \leqq j \leqq n$) is a linear substitution with* $\det(a_{ij}) \not\equiv 0 \pmod p$, *then the resulting polynomial $g(y_1, \dots, y_n)$ is again a permutation polynomial* mod p.

Proof. This simply follows from the fact that a linear substitution of the above form transforms a given M_p^n into another M_p^n.

Definition 2. Let Z_p denote the residue class ring $Z/(p)$. For $f \in Z[x_1, \dots, x_n]$, let $\bar{f}$ be the image of f under the canonical homomorphism from $Z[x_1, \dots, x_n]$ into $Z_p[x_1, \dots, x_n]$. Two polynomials $f, g \in Z[x_1, \dots, x_n]$ are said to be equivalent mod p if there exists a linear substitution T of the form mentioned in Lemma 2 such that $\overline{Tf} = \bar{g}$.

Equivalence mod p is easily seen to be an equivalence relation in $Z[x_1, \dots, x_n]$.

Lemma 3. *Let f be equivalent* mod p *to g; $f, g \in Z[x_1, \dots, x_n]$. Then f is a permutation polynomial* mod p *if and only if g is one.*

Proof. This follows from Lemma 2 and Definition 2.

4. Linear polynomials.

Theorem 3. *$f(x_1, \dots, x_n) = b_1x_1 + \dots + b_nx_n + b \in Z[x_1, \dots, x_n]$ is a permutation polynomial* mod p *if and only if* g.c.d. $(b_1, \dots, b_n, p) = 1$.

Proof. If g.c.d. $(b_1, \ldots, b_n, p)=p$, then $f(a_1, \ldots, a_n) \equiv b \pmod p$ for all lattice points and so f is not a permutation polynomial mod p. If g.c.d. $(b_1, \ldots, b_n, p)=1$, then WLOG g.c.d. $(b_n, p)=1$. But then $b_n x_n$ is a permutation polynomial mod p and so we can infer from Lemma 1 that f itself is one.

5. Quadratic polynomials, case $p \neq 2$.

Theorem 4. *Let $f \in Z[x_1, \ldots, x_n]$ be a polynomial of degree 2. Then f is a permutation polynomial* mod p *if and only if f is equivalent* mod p *to a polynomial of the form $g(x_1, \ldots, x_n) = h(x_1, \ldots, x_{n-1}) + b_n x_n$ with $h \in Z[x_1, \ldots, x_{n-1}]$,* g.c.d. $(b_n, p)=1$.

Proof. The sufficiency of the condition follows from Lemma 1 and Lemma 3 and the fact that $b_n x_n$ is a permutation polynomial mod p.

Conversely, suppose that f is a permutation polynomial mod p. Since Z_p is a field of characteristic $p \neq 2$, f is equivalent mod p to a polynomial of the form $r(x_1, \ldots, x_n) = u_1 x_1^2 + \cdots + u_k x_k^2 + d_{k+1} x_{k+1} + \cdots + d_n x_n + d$, $0 \leqq k \leqq n$, g.c.d. $(u_i, p)=1$ for $1 \leqq i \leqq k$. If $k<n$ and g.c.d. $(d_j, p)=1$ for at least one j, $k+1 \leqq j \leqq n$, then we are done. Otherwise, f is equivalent mod p to $s(x_1, \ldots, x_n) = u_1 x_1^2 + \cdots + u_k x_k^2 + d$, $0<k \leqq n$. By Lemma 3, s is a permutation polynomial mod p. On the other hand, we have for $m = 1, \ldots, p-1$:

$$\sum_{(a_1,\ldots,a_n) \in M_p^n} e^{2\pi i \frac{m}{p} s(a_1,\ldots,a_n)} = p^{n-k} e^{2\pi i \frac{m}{p} d} \left(\sum_{a_1=0}^{p-1} e^{2\pi i \frac{m}{p} u_1 a_1^2} \right) \cdots \left(\sum_{a_k=0}^{p-1} e^{2\pi i \frac{m}{p} u_k a_k^2} \right) =$$

$$= p^{n-k} e^{2\pi i \frac{m}{p} d} \sigma_1 \ldots \sigma_k \quad \text{with} \quad \sigma_j = \sum_{a_j=0}^{p-1} e^{2\pi i \frac{m}{p} u_j a_j^2}, \qquad 1 \leqq j \leqq k.$$

If $m u_j$ is a quadratic residue modulo p, then $\sigma_j = \sum_{a=0}^{p-1} e^{2\pi i \frac{a^2}{p}}$ and thus $|\sigma_j| = \sqrt{p}$ ([6], ch. 2). If $m u_j$ is a quadratic nonresidue modulo p, then $\sigma_j + \sum_{a=0}^{p-1} e^{2\pi i \frac{a^2}{p}} = 2 \sum_{b=0}^{p-1} e^{2\pi i \frac{b}{p}} = 0$ and thus $|\sigma_j| = \sqrt{p}$. In any case we have $\sigma_j \neq 0$ for all $j=1, \ldots, k$ and this contradiction to Theorem 2 completes the proof.

From a close inspection of the preceding proof we are led to a simple and systematic method for detecting quadratic permutation polynomials which is based on coefficient matrices. To fix this idea, we give the following definitions:

Definition 3. Let A be a matrix whose elements are rational numbers of the form a/b with $p \nmid b$. Then $\operatorname{rank}_p A$ is the rank of A, considered as a matrix over Z_p.

Definition 4. Let $f(x_1, \dots, x_n) = \sum_{1 \leqq i \leqq j \leqq n} a_{ij} x_i x_j + \sum_{r=1}^{n} c_r x_r + c$ be a quadratic polynomial from $Z[x_1, \dots, x_n]$. Then

$$A(f) = \begin{pmatrix} a_{11} & \frac{1}{2} a_{12} \cdots & \frac{1}{2} a_{1n} \\ \frac{1}{2} a_{12} & a_{22} \cdots & \frac{1}{2} a_{2n} \\ \vdots & \vdots & \vdots \\ \frac{1}{2} a_{1n} & \frac{1}{2} a_{2n} \cdots & a_{nn} \end{pmatrix}, \qquad A'(f) = \left(A(f) \left| \begin{matrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{matrix} \right. \right).$$

Let us note that the k in the proof of the preceding theorem is nothing else than $\operatorname{rank}_p A(f)$. Furthermore, f will be equivalent $\bmod p$ to a polynomial of the form given in Theorem 4 if and only if the last column of the augmented matrix $A'(f)$, considered as a vector over Z_p, is linearly independent of the preceding column vectors. Therefore:

Theorem 5. *A quadratic polynomial $f \in Z[x_1, \dots, x_n]$ is a permutation polynomial* $\bmod p$ *if and only if* $\operatorname{rank}_p A'(f) > \operatorname{rank}_p A(f)$.

6. Quadratic polynomials, case $p=2$. Since $a^2 \equiv a \pmod 2$ for integers a, we can replace terms x_i^2 by x_i whenever they occur. Having this convention in mind, we can prove the following

Theorem 6. *A polynomial $f \in Z[x_1, \dots, x_n]$ of degree* 2 *is a permutation polynomial* mod 2 *if and only if f is equivalent* mod 2 *to a polynomial of the form* $g(x_1, \dots, x_n) = h(x_1, \dots, x_{n-1}) + x_n$, $h \in Z[x_1, \dots, x_{n-1}]$.

Proof. The sufficiency of the condition follows from Lemma 1 and Lemma 3. Conversely, suppose that f is a permutation polynomial mod 2 whose degree modulo 2 is two (otherwise Theorem 3 yields the desired result). By possibly renaming the variables, we get modulo 2:

$$f(x_1, \dots, x_n) = x_1(x_{i_1} + x_{i_2} + \cdots + x_{i_d} + b) + f_1(x_2, \dots, x_n)$$

with $2 \leqq i_1 < i_2 < \cdots < i_d \leqq n$. Thus f is equivalent mod 2 to $x_1 x_2 + r(x_2, \dots, x_n)$. Consider $r(x_2, \dots, x_n)$ modulo 2. Let M be the least integer such that a term of the form $x_M x_j$, $M < j$, occurs in r, or $M = n+1$ if r is linear. If r contains a linear term x_i with $3 \leqq i < M$, then we are done. Otherwise, f is equivalent mod 2 to $x_1 x_2 + cx_2 + s(x_M, \dots, x_n)$. If $M=2$, then we apply the above reduction process to s and we get f equivalent mod 2 to $x_1 x_2 + x_2 x_3 + t(x_3, \dots, x_n)$ which, in turn, is equivalent mod 2 to $x_1 x_2 + t(x_3, \dots, x_n)$. Since this is also true for $M > 2$, we obtain by repeated application of the reduction process: f is either equivalent mod 2 to the desired form or, after possibly renaming the variables, to a polynomial of the form $q(x_1, \ldots, x_n) = x_1 x_2 + x_3 x_4 + \cdots + x_{2k-1} x_{2k}$.

We complete the proof by showing that q cannot be a permutation polynomial mod 2. In fact, using Theorem 2 with $m=1$, we have:

$$\sum_{(a_1,\ldots,a_n)\in M_2^n} e^{\pi i q(a_1,\ldots,a_n)} = 2^{n-2k}\left(\sum_{a_1=0}^{1}\sum_{a_2=0}^{1}(-1)^{a_1 a_2}\right)\cdots\left(\sum_{a_{2k-1}=0}^{1}\sum_{a_{2k}=0}^{1}(-1)^{a_{2k-1}a_{2k}}\right) \neq 0.$$

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Über Untergruppen mit ausgezeichneten Repräsentantensystemen

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1. Einleitung

Die vorliegende Arbeit, der eine Dissertation [1] über ausgezeichnete Untergruppen zugrunde liegt, untersucht einige Eigenschaften der folgenden Begriffe.

1. 1. Definition. Sei G eine Gruppe und $U \leqq H \leqq G$. Ein Repräsentantensystem R der Rechtsrestklassen von H in G heiße (U, H, G)-*System*, wenn R unter U invariant ist, d.h. wenn $u^{-1}Ru=R$ gilt für alle $u \in U$. Ein (H, H, G)-System heißt nach R. KOCHENDÖRFFER [4] ein *ausgezeichnetes Repräsentantensystem von H in G.* Wir nennen die Untergruppe H *ausgezeichnet in G*, wenn sie ein ausgezeichnetes Repräsentantensystem besitzt.

Es sei bemerkt, daß eine Teilmenge R von G genau dann ein (U, H, G)-System ist, wenn die Menge $L=\{r^{-1}|r \in R\}$ ein unter U invariantes Repräsentantensystem der Linksrestklassen von H in G ist. Man erhält also denselben Begriff der ausgezeichneten Untergruppe, wenn man zur Definition Links- statt Rechtsrestklassen benutzt.

Mit Hilfe eines in Kiel entwickelten Programmsystems [2] wurden für eine größere Anzahl von Gruppen alle ausgezeichneten Untergruppen berechnet. Diese Computer-Protokolle benutzte ich zum Aufsuchen von Gegenbeispielen, insbesondere in Abschnitt 3 bei der Frage nach Vererblichkeit der Eigenschaft, ausgezeichnete Untergruppe zu sein.

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2. Existenzkriterien

2. 1. Satz. *Sei G eine Gruppe, $U \leqq H \leqq G$, und sei S ein Repräsentantensystem der Doppelrestklassen HgU von G nach H und U. Dann sind folgende Aussagen gleichwertig.*

(1) *Es gibt ein* (U, H, G)*-System.*
(2) $g \in HC_G(U \cap H^g)$ *für alle* $g \in G$.
(3) $s \in HC_G(U \cap H^s)$ *für alle* $s \in S$.
(4) *Zu jedem* $g \in G$ *gibt es ein Element* $h \in H$, *so daß* $x^h = x^g$ *für alle* $x \in U^{g^{-1}} \cap H$.
(5) *Zu jedem* $s \in S$ *gibt es ein Element* $h \in H$, *so daß* $x^h = x^s$ *für alle* $x \in U^{s^{-1}} \cap H$.

Zum Beweis von 2. 1 benutzen wir zwei Hilfssätze, die sich durch Verallgemeinerung zweier Sätze von G. ZAPPA ([15, 16] Lemma 1 und 2) ergeben.

2. 2. Hilfssatz. *Sei G eine Gruppe und* $U \leqq H \leqq G$. *Sei R ein* (U, H, G)*-System nnd* $r \in R$. *Dann gilt* $r \in C_G(U \cap H^r)$.

Beweis. Sei $x \in U \cap r^{-1}Hr$. Dann ist einerseits $x^{-1}rx \in x^{-1}rr^{-1}Hr = Hr$ und andererseits $x^{-1}rx \in x^{-1}Rx = R$. Daraus folgt $x^{-1}rx = r$. Q.e.d.

2. 3. Hilfssatz. *Sei G eine Gruppe,* $U \leqq H \leqq G$, *und sei S ein Repräsentantensystem der Doppelrestklassen HgU von G nach H und U. Gibt es zu jedem* $s \in S$ *ein Element* $s^* \in HsU$ *mit der Eigenschaft* $s^* \in C_G(U \cap H^{s^*})$, *so bildet die Menge der Elemente* s^* *und ihrer Konjugierten unter U ein* (U, H, G)*-System.*

Ein Beweis für 2. 3 läßt sich bis auf geringfügige Änderungen aus [15] übernehmen, wo der Satz für den Spezialfall $U = H$ bewiesen wird.

Beweis von 2. 1. (1)→(2). Sei R ein (U, H, G)-System und $g \in G$. Dann gibt es ein $r \in R$ mit $g \in Hr$. Nach 2. 2 ist $r \in C_G(U \cap H^r)$. Wegen $r^{-1}Hr = g^{-1}Hg$ folgt $r \in C_G(U \cap H^g)$, also $g \in HC_G(U \cap H^g)$.

(2)→(4). Sei $g \in G$. Nach Voraussetzung läßt sich g schreiben als Produkt $g = hc$ mit $h \in H$ und $c \in C_G(U \cap H^g)$. Sei $x \in U^{g^{-1}} \cap H$. Dann ist $x^g \in U \cap H^g$, also $x^h = x^{gc^{-1}} = x^g$.

(4)→(5). Trivial.

(5)→(3). Wir setzen $h^{-1}s = c$. Sei $y \in U \cap H^s$. Dann ist $y^{s^{-1}} \in U^{s^{-1}} \cap H$, also $y = y^{s^{-1}s} = y^{s^{-1}h} = y^{c^{-1}}$. Daraus folgt $c \in C_G(U \cap H^s)$. Wegen $h \in H$ ist daher $s = hc \in$ $\in HC_G(U \cap H^s)$.

(3)→(1). Sei $s \in S$. Nach Voraussetzung läßt sich s schreiben als Produkt $s = hs^*$ mit $h \in H$ und $s^* \in C_G(U \cap H^s) = C_G(U \cap H^{s^*})$. Dabei ist $s^* = h^{-1}s \in HsU$. Nach 2. 3 gibt es daher ein (U, H, G)-System. Q.e.d.

Setzt man $U = H$, so liefert 2. 1 notwendige und hinreichende Kriterien für die Existenz eines ausgezeichneten Repräsentantensystems von H in G. Durch Hinzunahme weiterer Voraussetzungen erhält man insbesondere die folgenden einfachen Sonderfälle von Kriterium (2).

2. 4. Korollar. *Ein Normalteiler H einer Gruppe G ist genau dann ausgezeichnet in G, wenn* $G = HC_G(H)$ *ist.*

2.5. Korollar. *Eine abelsche Untergruppe H einer Gruppe G ist genau dann ausgezeichnet in G, wenn $g \in C_G(H \cap H^g)$ gilt für alle $g \in G$.*

2.6. Korollar. *Eine minimale Untergruppe H einer Gruppe G ist genau dann ausgezeichnet in G, wenn $N_G(H) = C_G(H)$ ist.*

2.7. Korollar. *Ein abelscher Normalteiler H einer Gruppe G ist genau dann ausgezeichnet in G, wenn er im Zentrum von G liegt.*

Nach C. H. Sah [13] heißt eine Untergruppe H einer Gruppe G c-abgeschlossen in G, wenn je zwei Elemente von H, die unter G konjugiert sind, auch schon unter H konjugiert sind. Aus Kriterium (4) von 2.1 folgt:

2.8. Korollar. *Sei G eine Gruppe und $H \leqq G$. Ist H ausgezeichnet in G, so ist H -abgeschlossen in G.*

Die Umkehrung von 2.8 gilt nicht. Ein Gegenbeispiel ist in [1] angegeben.

Besitzt eine Untergruppe in einer Gruppe G ein normales Komplement, so ist sie offensichtlich ausgezeichnet in G. Mit der Frage, wann umgekehrt zu einer ausgezeichneten Untergruppe ein normales Komplement existiert, beschäftigen sich verschiedene Arbeiten [5, 6, 9—12, 14—16]. Ein Ergebnis dieser Untersuchungen ist, daß eine ausgezeichnete Hallgruppe einer endlichen Gruppe ein normales Komplement besitzt, wenn sie einen Sylowturm hat. Dies folgt wegen 2.8 aus auch einem Satz von C. H. Sah ([13] Theorem 3), nach dem gilt: Ist H eine Hallgruppe einer endlichen Gruppe G und besitzt H einem Sylowturm oder einen nilpotenten Normalteiler mit nilpotenter Faktorgruppe, so existiert genau dann ein normales Komplement von H in G, wenn H in G c-abgeschlossen ist.

3. Einige Eigenschaften ausgezeichneter Untergruppen

In diesem Abschnitt diskutieren wir die Vererblichkeit der Eigenschaft, ausgezeichnete Untergruppe zu sein, auf gewisse Untergruppen, Faktorgruppen, Durchschnitte und Erzeugnisse. Die Beweise der Ergebnisse, die hier nur referiert werden, sind in [1] angegeben.

3.1. Satz. *Sei G eine Gruppe, $H \leqq K \leqq G$ und $H \leqq L \leqq G$. Dann gilt:*

3.1.1. (Prohaska [11] 2.1.) *Ist H ausgezeichnet in G, so ist H auch ausgezeichnet in K.*

3.1.2. (Zappa [14].) *Ist H ausgezeichnet in K und K ausgezeichnet in G, so ist H ausgezeichnet in G.*

3. 1. 3. *Sei $H \trianglelefteq K$ und H ausgezeichnet in K und in L. Ist $H \trianglelefteq L$ oder $\langle K, L\rangle = KL$, so ist H auch ausgezeichnet in $\langle K, L\rangle$.*

Aus 3. 1. 1 folgt unmittelbar: Ist H ausgezeichnet in zwei Untergruppen K und L von G, so ist H auch ausgezeichnet in ihrem Durchschnitt $K \cap L$. Eine entsprechende Aussage für das Erzeugnis $\langle K, L\rangle$ gilt nicht. Sei nämlich $G = \langle a, b, c, d\rangle$ die durch die definierenden Relationen $a^4 = 1$, $b^2 = a^2$, $c^3 = 1$, $d^2 = a^2$, $[a, b] = a^2$, $[a, c] = ba^2$, $[a, d] = b$, $[b, c] = ba^3$, $[b, d] = a^2$ und $[c, d] = c$ gegebene Gruppe der Ordnung 48, und seien $H = \langle d\rangle$, $K = \langle c, d\rangle$ und $L = K^b$. Dann ist H ausgezeichnet in K und in L, aber nicht in $\langle K, L\rangle$.

Auch einige andere wünschenswerte Eigenschaften besitzen die ausgezeichneten Untergruppen nicht. Sei zum Beispiel $G = \langle a, b, c, d\rangle$ die durch die definierenden Relationen $a^4 = 1$, $b^2 = a^2$, $c^2 = 1$, $d^2 = a^2$, $[a, b] = a^2$, $[a, c] = [a, d] = [b, c] = [b, d] = 1$ und $[c, d] = a^2$ gegebene Gruppe der Ordnung 32. Dann sind die Untergruppen $H = \langle a, b\rangle$ und $K = \langle a, cb\rangle$ normal und ausgezeichnet in G, aber $H \cap K$ und $\langle H, K\rangle$ sind nicht ausgezeichnet in G. Es bilden also im allgemeinen weder die ausgezeichneten Untergruppen noch die ausgezeichneten Normalteiler einer Gruppe G einen Teilverband des Untergruppenverbandes von G.

Ist weiterhin H eine ausgezeichnete Untergruppe einer Gruppe G und φ ein Homomorphismus von G, so braucht $H\varphi$ in $G\varphi$ nicht ausgezeichnet zu sein. Sei etwa $G = \langle a, b, c\rangle$ die durch die definierenden Relationen $a^2 = b^4 = c^2 = [a, b] = [a, c] = 1$ und $[b, c] = b^2 a$ gegebene Gruppe der Ordnung 16, und sei $N = \langle a\rangle$ und $H = \langle b\rangle$. Dann ist $N \triangleleft G$ und H ausgezeichnet in G, aber NH/N ist nicht ausgezeichnet in G/N. Es gilt jedoch der folgende Satz.

3. 2. Satz. *Sei G eine Gruppe, $N \trianglelefteq G$, $H \leqq G$ und H ausgezeichnet in G. Dann gilt:*

3. 2. 1. *Ist $N \leqq H$, so ist H/N ausgezeichnet in G/N.*

3. 2. 2. *Ist $H \trianglelefteq G$, so ist NH/N ausgezeichnet in G/N. (Außerdem ist dann $C_G(H)$ ausgezeichnet in G und $Z(H) \leqq Z(G)$).*

3. 2. 3. *Sei NH endlich und $|H|$ teilerfremd zu $N : N \cap H$. Jede Untergruppe von NH, deren Ordnung $|H|$ teilt, liege in einer unter NH zu H konjugierten Untergruppe (das ist insbesondere erfüllt, wenn NH auflösbar ist). Dann ist NH/N ausgezeichnet in G/N.*

Die Aussagen 3. 1. 1, 3. 1. 2 und 3. 2. 1 sind Sonderfälle der beiden folgenden Sätze.

3. 3. Satz. *Sei G eine Gruppe, $V \leqq K \leqq G$ und $U \leqq H \leqq K$. Dann gilt:*

3. 3. 1. *Ist R ein (U, H, G)-System, so ist $R \cap K$ ein (U, H, K)-System.*

3. 3. 2. *Ist R ein (U, H, K)-System und S ein (V, K, G)-System, so ist RS ein $(U \cap V, H, G)$-System.*

3. 4. Satz. *Sei G eine Gruppe, $U \leqq H \leqq G$, $N \leqq H$ und $N \trianglelefteq G$. Ist R ein (U, H, G)-System, so ist die Menge $\{Nr | r \in R\}$ ein $(NU/N, H/N, G/N)$-System.*

4. Ausgezeichnete Untergruppen von Primzahlordnung

In [11] stellt L. PROHASKA die Frage nach der Existenz endlicher auflösbarer Gruppen, in denen keine nicht triviale Untergruppe ausgezeichnet ist. Wir zeigen, daß jede endliche Gruppe, die eine nicht triviale Untergruppe besitzt, auch eine nicht triviale ausgezeichnete Untergruppe besitzt.

4. 1. Satz. *Sei G eine endliche Gruppe der Ordnung $|G| > 1$ und p die kleinste Primzahl, die $|G|$ teilt. Dann ist in G jede Untergruppe der Ordnung p ausgezeichnet.*

Beweis. Sei $H \leqq G$ mit $|H| = p$, und sei $g \in N_G(H)$. Dann induziert g auf H einen Automorphismus φ. Da die Ordnung der Automorphismengruppe von H gleich $p-1$ ist, ist der von g^{p-1} auf H induzierte Automorphismus φ^{p-1} die Identität, also $g^{p-1} \in C_G(H)$. Wegen $(|G|, p-1) = 1$ folgt hieraus $g \in C_G(H)$. Es ist daher $N_G(H) = C_G(H)$. Nach 2. 6 ist H ausgezeichnet in G. Q.e.d.

Für nilpotente Gruppen läßt sich die Aussage von 4. 1 verschärfen.

4. 2. Satz. *In einer endlichen nilpotenten Gruppe ist jede Untergruppe von Primzahlordnung ausgezeichnet.*

Beweis. Sei G eine endliche nilpotente Gruppe, $H \leqq G$ und $|H| = p$ eine Primzahl. Ist P die p-Sylowgruppe von G, so gilt $H \leqq P \leqq G$. Nach 4. 1 ist H ausgezeichnet in P, und als direkter Faktor besitzt P ein normales Komplement und damit ein ausgezeichnetes Repräsentantensystem in G. Daraus folgt nach 3. 1. 2 die Behauptung. Q.e.d.

Nicht jede endliche Gruppe, in der sämtliche Untergruppen von Primzahlordnung ausgezeichnet sind, ist nilpotent. Ein Gegenbeispiel ist etwa die alternierende Gruppe vom Grade 4. Es gilt jedoch der folgende Satz.

4. 3. Satz. *Eine endliche überauflösbare Gruppe G, in der sämtliche Untergruppen von Primzahlordnung ausgezeichnet sind, ist nilpotent.*

Zum Beweis von 4. 3 gebrauchen wir einen Hilfssatz.

4. 4. Hilfssatz. *Sei p eine Primzahl. Hat eine endliche auflösbare Gruppe G die Eigenschaft, daß alle Untergruppen der Ordnung p in ihr ausgezeichnet sind, so haben auch alle Untergruppen und Faktorgruppen von G diese Eigenschaft.*

Beweis. Für die Untergruppen von G folgt die Behauptung unmittelbar aus 3. 1. 1, für die Faktorgruppen beweisen wir sie durch Induktion nach der Ordnung von G. Es genügt zu zeigen: Ist M ein minimaler Normalteiler von G und $M < H \leqq G$ mit $H:M = p$, so ist H/M ausgezeichnet in G/M.

Ist $N_G(H) = N < G$, so ist nach Induktionsannahme H/M ausgezeichnet in N/M. Mit 2. 6 folgt daraus $C_{G/M}(H/M) = C_{N/M}(H/M) = N_{N/M}(H/M) = N_{G/M}(H/M)$, das heißt, H/M ist auch ausgezeichnet in G/M. Es sei daher im folgenden $H \trianglelefteq G$.

Gibt es in G zwei verschiedene maximale Untergruppen U und V, die H enthalten, so folgt aus 2. 7 und der Induktionsannahme $H/M \leqq Z(U/M)$ und $H/M \leqq Z(V/M)$. Dann ist aber $H/M \leqq Z(\langle U/M, V/M \rangle) = Z(G/M)$ und daher H/M ausgezeichnet in G/M. Wir können daher annehmen, daß G/H zyklisch von Primzahlpotenzordnung ist, etwa $|G/H| = q^r$. Wegen 4. 1 brauchen wir nur den Fall $q < p$ zu betrachten.

Ist $(|M|, p) = 1$, so gibt es in G eine nach Voraussetzung ausgezeichnete p-Sylowgruppe P der Ordnung p. Aus 2. 6 folgt $C_G(P) = N_G(P)$. Nach einem Satz von W. Burnside ([3] IV. 2. 6) besitzt daher P in G ein normales Komplement K. Dann ist K/M ein normales Komplement und damit ein ausgezeichnetes Repräsentantensystem von H/M in G/M.

Gilt andererseits $p \mid |M|$, so ist M als minimaler Normalteiler der auflösbaren Gruppe G eine elementarabelsche p-Gruppe. Sei $|M| = p^n$. Dann ist $|H| = p^{n+1}$. Wegen $\langle 1 \rangle \neq M \cap Z(H) \trianglelefteq G$ und $H:Z(H) \neq p$ ist $Z(H) = H$, also H abelsch.

Ist H elementarablesch, so gibt es in H genau p^n Untergruppen der Ordnung p, die nicht in M liegen. Die Menge dieser Untergruppen zerfällt unter G in vollständige Klassen Konjugierter, wobei die Länge jeder Klasse als Teiler von $G:H$ eine Potenz von q ist. Wegen $q \nmid p^n$ ist mindestens eine dieser Potenzen gleich 1. In G gibt es daher einen nach Voraussetzung ausgezeichneten Normalteiler P der Ordnung p mit $PM = H$. Nach 2. 7 gilt $P \leqq Z(G)$ und deshalb auch $H/M \leqq Z(G/M)$. Folglich ist H/M ausgezeichnet in G/M.

Ist H nicht elementarabelsch, so hat die Frattinigruppe $\Phi(H)$ die Ordnung p. Wegen $\Phi(H) \leqq M$ und $\Phi(H) \trianglelefteq G$ folgt $\Phi(H) = M$. Dann ist H eine zyklische Gruppe der Ordnung p^2, und M ist ausgezeichnet in G. Nach 2. 7 folgt $M \leqq Z(G)$. Es genügt wieder zu zeigen, daß auch $H/M \leqq Z(G/M)$ gilt.

Sei $H = \langle h \rangle$ und $g \in G$. Dann ist $g^{-1}hg \in H$, etwa $g^{-1}hg = h^s$. Wegen $h^p \in M \leqq$ $\leqq Z(G)$ gilt $h^p = g^{-1}h^p g = (h^p)^s$. Daraus folgt $s \equiv 1 \pmod{p}$. Sei etwa $s = tp + 1$. Dann ist $g^{-1}hg = (h^p)^t h \in Mh$ und daher $(Mg)^{-1} MhMg = Mh$. Q.e.d.

Beweis von 4. 3. Die Behauptung ist richtig für $|G| = 1$. Sei daher $|G| > 1$. Wir nehmen an, der Satz sei bereits bewiesen für alle überauflösbaren Gruppen kleinerer Ordnung, und zeigen, daß in G jede maximale Untergruppe normal ist.

Es seien H eine maximale Untergruppe und N ein minimaler Normalteiler von G. Die Faktorgruppe G/N erfüllt nach 4. 4 die Voraussetzungen des Satzes.

Ist $N \leqq H$, so gilt daher nach Induktion $H/N \lhd G/N$ und folglich $H \lhd G$. Ist andererseits $N \nleqq H$, so ist $N \cap H < N$. Es gilt $N \cap H \trianglelefteq H$ und, da N abelsch ist, auch $N \cap H \lhd N$. Daraus folgt $N \cap H \lhd \langle N, H \rangle = G$. Wegen der Minimalität von N ist dann $|N \cap H| = 1$. Als normales Komplement von H in G hat N die Ordnung $|N| = G:H$. Da G überauflösbar ist, ist $G:H$ eine Primzahl. Also ist N ausgezeichnet in G. Nach 2. 7 folgt $N \leqq Z(G) \leqq N_G(H)$ und damit $H \lhd \langle N, H \rangle = G$. Q.e.d.

5. Gruppen, die in jeder Obergruppe ausgezeichnet sind

In diesem Abschnitt bestimmen wir alle endlichen Gruppen, die aufgrund ihres Isomorphietyps in jeder Obergruppe ausgezeichnet sind.

5. 1. Hilfssatz. *Sei G eine Gruppe und $H \leqq G$. Ist $|H| \leqq 2$ oder H isomorph zur nicht abelschen Gruppe S_3 der Ordnung 6, so ist H ausgezeichnet in G.*

Beweis. Für $|H|=1$ ist die Behauptung trivial. Ist $|H|=2$, so ist $N_G(H) = C_G(H)$. Daraus folgt nach 2. 6 die Behauptung. Sei daher $H \cong S_3$. Wir benutzen Kriterium (4) von 2. 1.

a) Sei $g \in N_G(H)$. Dann induziert g auf H einen Automorphismus. Da alle Automorphismen von H innere Automorphismen sind, gibt es ein Element $h \in H$ mit $x^h = x^g$ für alle $x \in H = H^{g^{-1}} \cap H$.

b) Sei $g \notin N_G(H)$. Dann ist $H^{g^{-1}} \cap H < H$. Also ist $H^{g^{-1}} \cap H$ zyklisch. Sei etwa $H^{g^{-1}} \cap H = \langle a \rangle$. Es ist $a^g \in H$ und $|a^g| = |a|$. Da je zwei Elemente gleicher Ordnung von H unter H konjugiert sind, gibt es ein Element $h \in H$ mit $a^h = a^g$. Daraus folgt $x^h = x^g$ für alle $x \in \langle a \rangle = H^{g^{-1}} \cap H$. Q.e.d.

5. 2. Hilfssatz. *Sei K eine endliche Gruppe der Ordnung $|K| > 2$, die nicht zur nicht abelschen Gruppe S_3 der Ordnung 6 isomorph ist. Dann enthält die symmetrische Gruppe $S_{|K|}$ vom Grade $|K|$ eine zu K isomorphe Untergruppe, die in $S_{|K|}$ nicht ausgezeichnet ist.*

Beweis. Nach P. Hall ([7] S. 364—365, Beweis von Hilfssatz 1) enthält $S_{|K|}$ eine zu K isomorphe Untergruppe H mit der folgenden Eigenschaft: Ist $U \leqq H$ und ψ ein Automorphismus von U, so gibt es einen inneren Automorphismus von $S_{|K|}$, der jedes $u \in U$ auf $u\psi$ abbildet. Wir nehmen an, es sei $|H| > 2$ und H ausgezeichnet in $S_{|K|}$, und zeigen, daß dann $H \cong S_3$ ist.

a) Sei p eine Primzahl, P eine p-Sylowgruppe von H und φ ein Automorphismus von P. Dann gibt es ein Element $g \in S_{|K|}$ mit $u^g = u\varphi$ für alle $u \in P$. Wegen $P = P^{g^{-1}} \cap P \leqq H^{g^{-1}} \cap H$ gibt es nach Kriterium (4) von 2. 1 ein Element $h \in H$ mit $u^h = u^g = u\varphi$ für alle $u \in P$. Dabei ist $h \in N_H(P)$.

b) Wegen $|H|>2$ folgt aus a) insbesondere: H ist nicht abelsch.

Wir nehmen nun an, es sei $|\varphi|=p^s$ und $|h|=mp^t$ mit $(m, p)=1$. Dann gibt es zwei Elemente $h_1, h_2 \in \langle h \rangle$ mit $|h_1|=m$ und $|h_2|=p^t$, so daß $h=h_1 h_2$ ist. Wegen $h_1 \in \langle h^{p^t} \rangle \leqq \langle h^{p^s} \rangle \leqq C_H(P)$ gilt $u^{h_2}=u^h=u\varphi$ für alle $u \in P$. Insbesondere ist $h_2 \in N_H(P)$. Da P die einzige p-Sylowgruppe von $N_H(P)$ ist, folgt $h_2 \in P$. Also besitzt P keinen äußeren Automorphismus von p-Potenzordnung. Nach einem Satz von W. GASCHÜTZ ([3] III. 19. 1) ist dann $|P| \leqq p$. Daraus folgt: H hat quadratfreie Ordnung.

c) Sei nun p der größte Primteiler von $|H|$. Dann ist $|P|=p$, und P besitzt einen Automorphismus φ der Ordnung $p-1$. Nach a) gibt es ein Element $h \in H$ mit $u^h=u\varphi$ für alle $u \in P$. Also gilt $p-1 \mid |H|$. Wegen $|H|>2$ und $4 \nmid |H|$ ist p ungerade. Daraus folgt $2 \mid |H|$.

Wir müssen nur noch zeigen, daß $p=3$ ist. Wir nehmen an, es sei $p>3$. Da $p-1$ quadratfrei ist, gibt es dann eine ungerade Primzahl q, die $p-1$ teilt, und $\langle h \rangle$ enthält zwei Elemente x und y der Ordnungen $|x|=q$ und $|y|=2$. Sei $Q=\langle x \rangle$. Es ist $y \in C_H(Q)$. Da Q eine q-Sylowgruppe von H ist, gibt es nach a) ein Element $k \in H$ mit $x^k=x^{-1}$. Sei $z=k^{\frac{|k|}{2}}$. Dann ist $|z|=2$. Da $\frac{|k|}{2}$ ungerade ist, ist $x^z=x^{-1}$, also $z \in N_H(Q)$, aber $z \notin C_H(Q)$. Wegen $4 \nmid |H|$ sind $\langle y \rangle$ und $\langle z \rangle$ 2-Sylowgruppen von $N_H(Q)$, aber wegen $C_H(Q) \trianglelefteq N_H(Q)$ sind sie nicht konjugiert unter $N_H(Q)$. Die Annahme $p>3$ führt damit zu einem Widerspruch, und es ist $p=3$. Q.e.d.

Aus 5. 1 und 5. 2 folgt:

5. 3. Satz. *Eine endliche Gruppe hat genau dann die Eigenschaft, daß jede zu ihr isomorphe Untergruppe einer beliebigen Gruppe G in G ausgezeichnet ist, wenn sie von der Ordnung 1 oder 2 oder nicht abelsch und von der Ordnung 6 ist.*

6. Ausgezeichnete Untergruppen symmetrischer Gruppen

6. 1. Satz. *Sei Ω eine Menge und $\Delta \subseteqq \Omega$. Ist G die Gruppe aller Permutationen auf Ω und G_Δ die Untergruppe derjenigen Permutationen, die Δ elementweise fest lassen, so ist G_Δ ausgezeichnet in G.*

Beweis. Sei g ein beliebiges Element von G und $\Gamma=\{\delta g \mid \delta \in \Delta\}$. Dann ist $G_\Delta{}^g = g^{-1} G_\Delta g = G_\Gamma$ und daher $G_\Delta \cap G_\Delta{}^g = G_{\Delta \cup \Gamma}$. Sei $c \in G$ eine Permutation, die jedes $\delta \in \Delta$ auf δg abbildet und alle Elemente von $\Omega-(\Delta \cup \Gamma)$ fest läßt. Dann ist $c \in C_G(G_{\Delta \cup \Gamma}) = C_G(G_\Delta \cap G_\Delta{}^g)$. Da die Permutation gc^{-1} alle Elemente von Δ fest läßt, gilt außerdem $gc^{-1} \in G_\Delta$, also $g = gc^{-1} c \in G_\Delta C_G(G_\Delta \cap G_\Delta{}^g)$. Nach Kriterium (2) von 2. 1 ist daher G_Δ ausgezeichnet in G. Q.e.d.

Aus 6. 1 folgt insbesondere, daß es in jeder symmetrischen Gruppe S_n vom Grade $n \geqq 1$ zu jeder Zahl m mit $1 \leqq m \leqq n$ eine ausgezeichnete Untergruppe vom

Typ S_m gibt. Nach Computerberechnungen von W. LINDENBERG [8] sind für $n \leqq 6$ in der S_n sogar alle symmetrischen Untergruppen ausgezeichnet. Andererseits folgt aus 5. 2 mit Hilfe von 3. 1. 1, daß es in jeder S_n mit $n \geqq 24$ eine nicht ausgezeichnete Untergruppe vom Typ S_4 gibt.

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Rings whose proper subrings have property P

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Let D be an integral domain with identity. GILMER in [2] determines necessary and sufficient conditions in order that each subring of D with identity be Noetherian. We consider essentially the same type of problem. In general, let R be a ring (not necessarily commutative) and let P be a ring theoretic property. We seek to determine necessary and sufficient conditions on R in order that each proper subring of R has property P.

In this paper we will consider these two properties P:

(P1) Has finite characteristic.

(P2) Has no proper zero divisors.

It is clear that if R satisfies (P1) or (P2), then each proper subring of R satisfies (P1) or (P2), respectively, but neither of the converses is true. Corollary (2. 3) gives a characterization of rings for which each proper subring has finite characteristic, and Corollary (2. 11) characterizes rings for which each proper subring has no proper zero divisors. Moreover, Proposition (2. 5) and Corollary (2. 10) give necessary and sufficient conditions on R in order that each proper two-sided ideal of R satisfies (P1) or (P2), respectively, and Corollaries (2. 7) and (2. 12) give necessary and sufficient conditions on R in order that each proper left (right) ideal of R satisfies (P1) or (P2), respectively.

Section 1 contains the necessary notation and definitions used in the paper and includes the statement of one lemma, which we use frequently. Section 2 contains our main results. Throughout the paper the symbols $\subseteq$ and $\subset$ will denote containment, and proper containment, respectively. We will use the symbols Z and ω to denote the sets of integers and positive integers, respectively. The authors hereby express their appreciation to CRAIG WOOD for helpful comments concerning this paper. In particular, Wood suggested that the authors work on the problem of finite characteristic.

1. Preliminaries. All rings considered in this paper are assumed to contain more than one element. Throughout the paper, *ideal* will always mean two-sided ideal. If R is a ring and if $\{x_\alpha\}$ is a collection of elements of R, then $[\{x_\alpha\}]$ will denote the subring of R, and $(\{x_\alpha\})$ the ideal of R, generated by $\{x_\alpha\}$. If A is a subring (or ideal) of the ring R, then, following [3; p. 2], we say that A is *genuine* if $A \neq R$, and *proper* if A is genuine and nonzero.

If x is an element of a ring R and if there exists a positive integer n such that $n \cdot x = 0$, then the minimal positive integer for which this is true is called the *order* of x. If no such positive integer exists, we say that x has *infinite order*. If there exists a positive integer n such that $n \cdot x = 0$ for all $x \in R$, the smallest such positive integer is called the *characteristic* of R. If no such positive integer exists, we say that R has *characteristic zero*. If $x \cdot y = 0$ for each $x, y \in R$, we will say that R is the *zero ring* on R^+, the additive group of R; we will also say in this case that R has the *trivial multiplication*.

We will say that an element x of R is a proper *zero divisor* of R if x is nonzero and if there exists a nonzero element a of R such that either xa or ax is zero. A ring R is said to be *simple* if its only (two-sided) ideals are R and (0).

The following lemma is used frequently. It appears as an exercise in [4; p. 101].

(1. 1) Lemma. *If R is a ring with the property that the only left (right) ideals of R are R and* (0), *then R is either:*

(i) *a division ring, or*

(ii) *the zero ring on a finite cyclic group of prime order.*

2. Properties (P1) **and** (P2). Our first concern will be to characterize rings R with the property that each proper subring of R satisfies (P1).

As we have already observed, the condition that R have finite characteristic is not necessary. A zero ring on a quasicyclic group is an example of a ring R with characteristic zero such that each proper subring of R satisfies (P1). We will show that these are the only rings of characteristic zero with the property that each proper subring satisfies (P1).

Before proving the above statement, we give a brief description of a quasicyclic group. (See, for example, [1], [5], [6].) Let be a prime integer. The p-quasicyclic group, which we denote by $C(p^\infty)$, is a group generated by a set $\{c_i\}_{i\in\omega}$ such that c_i has order p^i, and $pc_{i+1} = c_i$, for each $i \in \omega$; there is, to within isomorphism, exactly one group with these properties. (The group of all complex pth power roots of unity, under multiplication, is a realization of $C(p^\infty)$.) The proper subgroups of $C(p^\infty)$ are exactly the finite cyclic groups generated by the c_i's. Thus, with the trivial multiplication, the proper subrings (and proper ideals) of $C(p^\infty)$ are just the proper subgroups of $C(p^\infty)$. It then follows that each proper subring (ideal) of $C(p^\infty)$ has finite characteristic, but $C(p^\infty)$ does not.

(2. 1) Theorem. *Let R be a ring such that every proper subring of R satisfies property* (P1). *R has finite characteristic if any of these five conditions hold:*

(i) *There exists a genuine ideal A of R such that R/A has finite characteristic.*
(ii) *R has an identity element.*
(iii) *R is commutative and contains a maximal ideal.*
(iv) *$R=[r]$ for some $r\in R$.*
(v) *There exist genuine ideals R_1, R_2 of R such that $R = R_1+R_2$.*

Proof. (i) By assumption $nR\subseteq A$ and $mA=(0)$ for some positive integers n and m. Hence $mnR=(0)$, and R has finite characteristic.

(ii) If e is the identity element of R, then e must have finite order, for otherwise $[2e]$ is a proper subring of R, and $[2e]$ does not have finite characteristic.

(iii) Let M be a maximal ideal of R. By Lemma (1. 1), R/M is either a field or R/M is finite. Part (i) of this theorem shows that R has finite characteristic if R/M is a field of finite characteristic or if R/M is finite. If R/M is a field of characteristic zero, choose $r\in R$ such that $r+M$ is the identity element of R/M. Then $nr\notin M$ and $nr^2\notin M$ for any positive integer n; in particular, $nr^2\neq 0$, so that $R=[r]=[r^2]$. Thus, $r=sr^2+mr^2=(sr+mr)r$ for some $s\in R$, $m\in Z$. Since $R=[r]$, it follows easily that $sr+mr$ is an identity element for R, and R has finite characteristic by (ii).

(iv) If $R=[r]$, then R is commutative and any ideal of R maximal with respect to not containing r (such ideals exist by a Zorn's Lemma argument) is a maximal ideal of R. Hence R has finite characteristic by (iii).

(v) If $R = R_1+R_2$, then $n_1R_1=0$ and $n_2R_2=0$ for some positive integers n_1 and n_2. Thus $n_1n_2R=(0)$, and R has finite characteristic.

(2. 2) Theorem. *Suppose that R is a ring such that*

(i) *each genuine ideal of R has finite characteristic,*
(ii) *R does not have finite characteristic, and*
(iii) *each element of R has finite order.*

Then R is the zero ring on a quasicyclic group.

Proof. Let $r\in R-\{0\}$ have order m, and let p be a prime divisor of m. Then $R_p=\{x\in R|p^nx=0 \text{ for some } n\in\omega\}$ and $S=\{x\in R|\text{the order of } x \text{ is not divisible by } p\}$ are ideals of R, and, as is well known, R is the direct sum of R_p and S. By choice of p, $R_p\neq(0)$; hence part (v) of Theorem (2. 1) implies that $S=(0)$ and $R=R_p$. Moreover, since nR is an ideal of R, it follows from (i) and (ii) that $R=nR$ for each positive integer n. Thus, if $x, y\in R$ and if y has order p^m, then there exists $a\in R$ such that $x=p^ma$. Therefore, $x\cdot y=(p^ma)y=a(p^my)=0$, so that R has the trivial multiplication.

We show that R is the zero ring on $C(p^\infty)$. Choose $d_1\in R$ such that d_1 has order p. Since $R=pR$, there exists $d_2\in R$ such that $pd_2=d_1$. Then d_2 has order p^2.

Inductively, choose $d_{i+1} \in R$ such that $pd_{i+1} = d_i$ for each $i \in \omega$. Then d_i has order p^i for each $i \in \omega$, and the ideal T generated by $\{d_i\}_{i \in \omega}$ has characteristic zero. Thus, $T = R$. But since R has the trivial multiplication, the ideal T is the same as the additive subgroup generated by $\{d_i\}_{i \in \omega}$. It follows that R is the zero ring on the p-quasicyclic group.

As a corollary to Theorems (2. 1) and (2. 2), we obtain the following result.

(2. 3) Corollary. *Let R be a ring. Each proper subring of R satisfies property* (P1) *if and only if one of the following conditions is satisfied:*

(i) *R has finite characteristic.*

(ii) *R is the zero ring on a quasicyclic group.*

Proof. We have already observed the sufficiency of conditions (i) and (ii). Conversely, if R does not have finite characteristic, then part (iv) of Theorem (2. 1) implies that r has finite order for each $r \in R$. Therefore, by Theorem (2. 2), R is the zero ring on a quasicyclic group.

Since the trivial multiplication defined on any abelian group G induces a ring structure on G, we have the following.

(2. 4) Corollary. (Compare with Ex. 23, p. 22 of [5].) *Let G be an abelian group. If each proper subgroup of G has bounded order, then either G has bounded order or G is a p-quasicyclic group.*

(2. 5) Proposition. *Let R be a ring. Each proper ideal of R satisfies property* (P1) *if and only if one of the following conditions is satisfied:*

(i) *R has finite characteristic.*

(ii) *R is the zero ring on a quasicyclic group.*

(iii) *R is a simple ring having no nonzero element of finite order.*

Proof. The sufficiency is obvious. Suppose that R does not have finite characteristic. If each element of R has finite order, then Theorem (2. 2) implies that R is the zero ring on a quasicyclic group. If there exists an element r of R of infinite order, and if A is the set of elements of R of finite order, then A is a genuine ideal of R, and, by hypothesis, there exists a positive integer k such that $kA = (0)$. However, since nR is an ideal of R, it follows from part (i) of Theorem (2. 1) that $R = nR$ for each positive integer n, and, in particular, $R = kR$. Thus, if $a \in A$, then $a = kr$ for some $r \in R$. But $ka = 0$ implies that $k^2 r = 0$, so that $r \in A$ and $a = kr = 0$. Therefore $A = (0)$, so that R is a simple ring with no nonzero element of finite order.

As a consequence of Proposition (2. 5) and its proof, we have the following result.

(2. 6) Corollary. *Let R be a simple ring of characteristic zero. Then R^+, the additive group of R, is isomorphic to a (weak) direct sum of full rational groups.*

Proof. Since R^+ is a divisible, torsion-free abelian group, the result follows from [1; Theorem 19. 1].

(2. 7) Corollary.[1]) *Let R be a ring. Each proper left (right) ideal of R satisfies property* (P1) *if and only if one of the following conditions is satisfied:*

(i) *R has finite characteristic.*

(ii) *R is the zero ring on a quasicyclic group.*

(iii) *R is a division ring of characteristic zero.*

Proof. If R has characteristic zero and if R is not the zero ring on a quasicyclic group, then Proposition (2. 5) implies that R is a simple ring with no nonzero element of finite order. Therefore, R has no proper left or right ideals, so that R is a division ring (Lemma (1. 1)).

We now turn our attention to property (P2). We use the following lemma; its proof is straightforward.

(2. 8) Lemma. *Let R be a ring containing no nonzero nilpotent element. If $a, b \in R$ and if $ab=0$, then $ba=0$, $axb=0$ for each x in R, and $bya=0$ for each y in R. Moreover, $Ra \cap Rb = aR \cap bR = RaR \cap RbR = (0)$.*

(2. 9) Theorem. *Let R be a ring such that R does not satisfy* (P2), *but each proper ideal of R satisfies* (P2).

(i) *If R contains no nonzero nilpotent element, then R is the direct sum of two simple rings, each of which satisfies property* (P2).

(ii) *If R contains a nonzero nilpotent element, then either:*

(a) *R is the zero ring on a cyclic group of prime order, or*

(b) *R is a simple ring such that $R^2=R$.*

Proof. (i) By assumption, there exist nonzero elements a and b of R such that $ab=0$. Since $a^3 \in RaR$ and $b^3 \in RbR$, RaR and RbR are nonzero ideals of R, so that $RaR+RbR = RaR \oplus RbR$ is a nonzero ideal of R that does not satisfy property (P2). Moreover, Lemma (2. 8) implies that RaR and RbR are proper ideals of R, and hence have property (P2). Let S be any nonzero ideal of the ring RaR. Then $RbR \cdot S = S \cdot RbR = (0)$, $S+RbR$ is an ideal of R, and hence $R = S+RbR$. By the modular law, $RaR = RaR \cap (S+RbR) = S+(RaR \cap RbR) = S+(0) = S$. Therefore, (0) and RaR are the only ideals of the ring RaR, so that RaR is a simple ring that satisfies property (P2). Similarly, RbR is a simple ring that satisfies property (P2).

(ii) Suppose that b is a nonzero element of R such that $b^2=0$. Then $(b)=R$. Let M be a genuine ideal of R. If $m \in M$, then bm and mb are elements of M, so that

[1]) The conditions which we obtain here, as well as those of Corollary (2. 12), are reminiscent of the conditions obtained by F. Szász in [7] characterizing rings in which every proper left ideal is cyclic.

$mb \cdot bm = 0$ implies that $bm=0$ or $mb=0$. In particular, $mbm=0$ for each $m \in M$. Thus, $(mb)^2=(bm)^2=0$, and hence $mb=bm=0$ for each $m \in M$. Since $(b)=R$, it follows that for each $r \in R$, $m \in M$, $rm=mr=0$. In particular, $M^2=(0)$, so that $M=(0)$ and R is a simple ring. If $R^2=0$, then R has trivial multiplication, and Lemma (1. 1) implies that R is the zero ring on a cyclic group of prime order. This completes the proof.

As an immediate consequence of Theorem (2. 9), we have the following.

(2. 10) Corollary. *Let R be a ring. Each proper ideal of R satisfies property* (P2) *if and only if one of the following conditions is satisfied:*

(i) *R satisfies property* (P2).

(ii) *R is the direct sum of two simple rings, each of which satisfies property* (P2).

(iii) *R is the zero ring on a cyclic group of prime order.*

(iv) *R does not satisfy* (P2) *and R is a simple ring for which $R^2=R$.*

We now use Theorem (2. 9) to obtain a characterization of rings for which each proper subring satisfies property (P2).

(2. 11) Corollary. *Let R be a ring. Each proper subring of R satisfies property* (P2) *if and only if one of the following conditions is satisfied:*

(i) *R satisfies property* (P2).

(ii) *$R \cong Z/(p) \oplus Z/(q)$, where p and q are prime integers.*

(iii) *R is the zero ring on a cyclic group of prime order.*

Proof. The sufficiency is obvious. Suppose that R does not satisfy property (P2). If x is a nonzero element of R with $x^2=0$, then $(0) \subset [x] = \{\lambda x | x \in Z\}$, and $[x]$ has trivial multiplication. Hence $[x]=R$, and Theorem (2. 9) implies that R is the zero ring on a cyclic group of prime order.

If R contains no nonzero nilpotent element, then Theorem (2. 9) implies that R is the direct sum of two simple rings, R_1 and R_2, each of which satisfies property (P2). Moreover, if S is a nonzero subring of R_1, then $S \cdot R_2 = (0)$, and $R = S + R_2 = R_1 \oplus R_2$. This imples that $S=R_1$, and R_1 is a ring with property (P2) having exactly two subrings R_1 and (0). It follows immediately from Lemma (1. 1) that R_1 is a finite prime field. Similarly, R_2 is a finite prime field, and $R \cong Z/(p) \oplus Z/(q)$, where p and q are prime integers.

(2. 12) Corollary. *Let R be a ring. Each proper left (right) ideal of R satisfies property* (P2) *if and only if one of the following conditions is satisfied:*

(i) *R satisfies property* (P2).

(ii) *R is the zero ring on a cyclic group of prime order.*

(iii) *R is the direct sum of two division rings.*

The proof is similar to that of Corollary (2. 11) and we omit it.

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Slender modules, slender rings

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Introduction. The notion of slenderness is due to J. ŁOŚ, who defined it originally for abelian groups in 1958 ([1], p. 169).

An abelian group is a Z-module, Z denoting the ring of integers, so that the theory of abelian groups appears as a special case of the theory of modules.

Conversely, a large part of the theory of abelian groups can be generalized to some classes of modules. The purpose of this paper is to indicate how the notion of slenderness can be extended in a natural way to modules and rings (considered as modules).

In a fundamental paper ([2], p. 71, Corollary 6), R. J. NUNKE has characterized all slender torsion free abelian groups: a torsion free abelian group is slender if and only if it contains no copy of the additive group Q of rational numbers, no copy of the additive group P of p-adic integers for any prime p, and no copy of the complete direct sum π of countably many infinite cyclic groups Z.

Similarly, the main problem in the theory of slender modules is to characterize all slender R-modules (for a given ring R), i.e. to investigate a theorem which may be the extension of Nunke's theorem to modules. It appears that such a generalization of Nunke's theorem to modules is not a trivial one and gives rise to hard problems of ring-theory and homological algebra. In this paper, we give only some *necessary* conditions for *any* module to be slender (Theorem 9), so that the problem of characterization of slender modules remains open. Finally, in § 4, we investigate some slender rings.

N.B. All rings considered in this paper are associative, commutative, with identity 1.

§ 1. Definition of slender modules and slender rings

1°) Let R be a ring. An R-module M is said to be *R-slender* (or, more briefly, *slender*) if for any non-zero homomorphism

$$h: \sum_{n=1}^{\infty}{}^{*} R_n \to M \qquad (R_n \cong R,\ n=1, 2, \dots)$$

we have $h(R_n)=0$ *for almost all* n.

2°) A ring R is said to be *slender* if R, considered itself as an R-module, is slender, i.e. if, for any non-zero homomorphism

$$h: \sum_{n=1}^{\infty}{}^{*} R_n \to R \qquad (R_n \cong R, \ n=1, 2, \ldots)$$

we have $h(R_n)=0$ *for almost all* n.

§ 2. Relation between slender modules over different rings

Theorem 1. *Let $R \subseteqq R'$ be two rings having the same identity 1. If an R'-module M is R-slender, then M is also R'-slender.*

Proof. Let the R'-module M be R-slender. Suppose that M is not R'-slender. Then there exists a non-zero homomorphism

$$h': \sum_{n=1}^{\infty}{}^{*} R'_n \to M \qquad (R'_n \cong R', \ n=1, 2, \ldots)$$

such that $h'(R'_n) \neq 0$ for infinitely many indices n. Then $h'(e_n)=h(e_n)\neq 0$ for infinitely many indices n, where

$$e_n = (\underbrace{0, \ldots, 0, 1}_{n}, 0, \ldots)$$

so that $h(R_n)\neq 0$ for infinitely many indices n, a contradiction to the hypothesis that M is R-slender. Hence M is R'-slender. Q.e.d.

Corollary 2. *A torsion free ring whose additive group is slender is slender itself.*

Proof. Let R' be a torsion free ring whose additive group is slender, $Z \subseteqq R'$. Since the Z-module R' is slender, from Theorem 1 we obtain that R'-module R' is slender. Q.e.d.

§ 3. Some necessary conditions for a module to be slender

Theorem 3. *Injective modules are not slender.*

Proof. Let M be an injective R-module. Consider the commutative diagram

$$\begin{tikzcd} & M & \\ 0 \arrow[r] & \sum_{n=1}^{\infty} R_n \arrow[u, "f"] \arrow[r, "j"] & \sum_{n=1}^{\infty}{}^{*} R_n \arrow[ul, dashed, "h"'] \end{tikzcd}$$

where f is the non-zero homomorphism defined by

$$f\left(\sum_{i=1}^{K} r_i e_i\right) \overset{\mathrm{df}}{=} \left(\sum_{i=1}^{K} r_i\right) \cdot m$$

with a fixed non-zero element m of M, $e_i = (\underbrace{0, \ldots, 0, 1}_{i}, 0, \ldots) \in R_i$, and $r_i \in R$ $(i=1, 2, \ldots)$. In particular, we have $f(e_i) = hj(e_i) = h(e_i) = m \neq 0$ for every $i = 1, 2, \ldots$. Hence M is not slender. Q.e.d.

Theorem 4. *Quotient modules of injective modules are not slender.*

Proof. Let M be an injective R-module, $N(\neq M)$ a submodule and $m \in M \setminus N$. Consider the commutative diagram

$$\begin{tikzcd} & M \arrow[r, "p"] & M/N \\ 0 \arrow[r] & \sum_{n=1}^{\infty} R_n \arrow[u, "f"] \arrow[r, "j"] & \sum_{n=1}^{\infty}{}^{*} R_n \arrow[ul, dashed, "h"'] \arrow[u, dashed, "g"'] \end{tikzcd}$$

where f is the non-zero homomorphism defined in the proof of Theorem 3. We have $g(e_i) = ph(e_i) = p(m) \neq 0$ for every $i = 1, 2, \ldots$. Hence M/N is not slender. Q.e.d.

Corollary 5. *Vector spaces are not slender.*

Proof. It is known that every module over a field (i.e. every vector space) is injective.

Corollary 6. *Fields are not slender.*

Theorem 7. *The ring P of p-adic integers is not slender.*

Proof. The homomorphism

$$\sum_{n=1}^{\infty}{}^{*} P_n \to P \qquad (P_n \cong P;\ n = 1, 2, \ldots)$$

defined by

$$(\pi_1, \pi_2, \ldots) \to \sum_i \pi_i p^i \in P$$

is such that

$$e_i = (\underbrace{0, \ldots, 0, 1}_{i}, 0, \ldots) \to p^i \neq 0$$

for every $i = 1, 2, \ldots$. Hence P is not slender. Q.e.d.

Theorem 8. *Let R be a ring. Then the complete direct sum $\sum_{n=1}^{\infty}{}^* R$ is not R-slender.*

Proof. The proof results from the simple fact that the identity transformation

$$\mathrm{id}: \sum_{n=1}^{\infty}{}^* R \to \sum_{n=1}^{\infty}{}^* R$$

is a homomorphism.

Now, we summarize the results of this paragraph. Since a submodule of a slender module is again slender, Theorem 3 and Theorem 8 give us the following *necessary* conditions for *any* module to be slender:

Theorem 9. *Let R be a ring. If an R-module M is R-slender, then it contains no copy of any injective R-module and no copy of the complete direct sum $\sum_{n=1}^{\infty}{}^* R$.*

§ 4. Some slender rings

Consider now the following problem:

Find all slender torsion free rings of rank 1.

It is known that every torsion free abelian group of rank 1, not isomorphic to the additive group Q of rational numbers, is slender. If we combine this result and Corollary 2, we obtain the solution of the above problem:

Theorem 10. *Every torsion free ring of rank 1, not isomorphic to the ring Q of rational numbers, is slender.*

The structure of all torsion free rings of rank 1 is completely known (cf. [3]), so that we obtain, in this way, an important class of slender rings. In particular, the ring $Z_{(p)} = \left\{\frac{k}{p^n}\right\}$ (p prime; $k \in Z$) and the ring Z of integers are slender.

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On random multiplicative functions

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1. We call $f(n)$ a completely multiplicative (c.m.) function, if $f(mn)=f(m)f(n)$ holds for all pairs m, n of positive integers. Let $\mathcal{F}$ be the set of those c. m. functions which take the values $+1$ and -1 only.

We say that a function $f(n)\in\mathcal{F}$ is of *normal type,* if

$$\lim_{x} x^{-1} N\{n\leqq x;\ f(n+i)=\varepsilon_i;\ i=0, \dots, k\} = \frac{1}{2^{k+1}} \tag{1.2}$$

for $k=0, 1, 2, \dots$ and for all choices of $\varepsilon_0=\pm 1, \dots;\ \varepsilon_k=\pm 1$.

It would be interesting to give a necessary and sufficient condition for $f(n)$ to be of normal type. Recently E. WIRSING [1] proved that a function $f(n)\in\mathcal{F}$ satisfies (1. 1) with $k=0$ if and only if

$$\sum_{f(p)=-1} \frac{1}{p} = \infty. \tag{1.2}$$

As is easy to see, the validity of (1. 2) is not sufficient for normality. Let for example $f(n)$ be defined as follows: $f(2)=1$, and for an odd prime p let $f(p)=1$ or -1 according as $p\equiv 1$ or -1 (mod 4). Then, by an easy calculation we have

$$\sum_{n\leqq x} f(n)f(n+4) = \frac{x}{4}+o(x);$$

hence it follows that $f(n)$ is not a normal function.

We shall see in the following section that almost all multiplicative functions are of normal type. One would think that the Liouville function $\lambda(n)$ is normal. However we can only prove that the system $\lambda(n)=\varepsilon_1$, $\lambda(n+1)=\varepsilon_2$ has infinitely many solutions for an arbitrary choice of $\varepsilon_1=\pm 1$, $\varepsilon_2=\pm 1$. This is a special case of the assertions which we shall prove in the section 3.

2. Let $c, c_1, c_2, \dots$ denote suitable positive constants; let $\varepsilon, \varepsilon_1, \varepsilon_2, \dots$ be arbitrary small positive constants not necessarily the same at every occurrence. Let $d_k(n)$ denote the number of solutions of the equation $n=x_1, \dots, x_k$ in positive integers $x_1, \dots, x_k$, and let $d_2(n)=d(n)$.

Let p_n denote the nth prime number. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\xi_n = \xi_n(\omega)$ $(n=1, 2, \ldots)$ be a sequence of independent random variables with the distribution $P(\xi_n = 1) = P(\xi_n = -1) = \frac{1}{2}$. Let $f(n; \omega)$ be a completely multiplicative function which we define on the set of primes by $f(p_n; \omega) = \xi_n(\omega)$.

We have

Theorem 1. *Almost all $f(n; \omega)$ are of normal type.*

For the proof we need some lemmas.

Lemma 1. *For positive integers C, D let $N(z; D, C)$ denote the number of solutions of the diophantine equation*

$$x^2 - Dy^2 = C \tag{2.1}$$

in positive integers x, y satisfying $x \leqq z$. Then

$$N(z; D, C) \leqq c_1 d(C^2) \log 2Dz. \tag{2.2}$$

Perhaps this lemma is known, but I was unable to find a reference to it. We prove now (2. 2).

Without any restriction we can assume that D is a square-free number. For $D=1$ inequality (2. 2) obviously holds, therefore we assume that $D>1$.

Let $K(\sqrt{D})$ denote the quadratic extension field over the rational number-field generated by $\sqrt{D}$. Let R denote the ring of the algebraic integers in $K(\sqrt{D})$, and for a general $\gamma \in R$ let (γ) denote the principal ideal generated by γ.

For a general solution x, y of (2. 1) let $\alpha = x + \sqrt{D}y$, $\beta = x - \sqrt{D}y$. Let $(C) = \pi_1^{\gamma_1} \ldots \pi_r^{\gamma_r}$, where $\pi_1, \ldots, \pi_r$ are different prime ideals. Using the fact that the norm of the ideals is a multiplicative functionand that $N((C)) = C^2$, furthermore that $N(\pi_i)$ is a prime number or a square of a prime number we have $\prod_{i=1}^{r} (\gamma_i + 1) \leqq d(C^2)$. Since $\alpha\beta = C$ and $\alpha, \beta \in R$, therefore $(\alpha)(\beta) = (C)$ and so $(\alpha) | (C)$. Hence it follows that all the solutions can be classified into at most $d(C^2)$ classes, where two solutions x, y; x_1, y_1 belong to the same class if and only if $(\alpha) = (x + \sqrt{D}y) = (\alpha_1) = (x_1 + \sqrt{D}y_1)$. Now we prove that the number of solutions of (2. 1) belonging to a fixed class does not exceed $c_1 \log 2Dz$, whence (2. 2) immediately shall follow.

Let (x_ν, y_ν) $\nu = 0, 1, \ldots, M$ be the all solutions in a class satisfying $1 \leqq x_0 \leqq x_1 \leqq \cdots \leqq x_M \leqq z$, $y_\nu \geqq 0$ and let $\alpha_\nu = x_\nu + y_\nu \sqrt{D}$, $\beta_\nu = x_\nu - y_\nu \sqrt{D}$. We have $(\alpha_0) = (\alpha_1) = \cdots = (\alpha_M)$. Therefore $\alpha_\nu = \alpha_\mu \varepsilon_{\nu\mu}$, $\beta_\nu = \beta_\mu \varrho_{\nu\mu}$, where $\varepsilon_{\nu\mu}$, $\varrho_{\nu\mu}$ are units in R. Since $C = \alpha_\nu \beta_\nu = \alpha_\mu \beta_\mu \varrho_{\nu\mu} \varepsilon_{\nu\mu} = \varrho_{\nu\mu} \varepsilon_{\nu\mu} C$, we have $\varrho_{\nu\mu} = \varepsilon_{\nu\mu}^{-1}$. Using the Dirichlet theorem concerning the form of the units we see that all units have form $\pm \varepsilon_0^n$ $(n = 0, \pm 1, \pm 2, \ldots)$, where $\varepsilon_0 = \frac{u_0 + \sqrt{D} v_0}{2}$, and u_0, v_0 are suitable positive integers satisfying $u_0^2 - Dv_0^2 = 4$. Hence $\varepsilon_0 > \frac{\sqrt{D}}{2}$ and we can assume that $\alpha_n = \alpha_0 \varepsilon^n$. Using that $x_n \leqq z$ and that by (2. 1) $y_n \leqq \sqrt{\frac{C+z^2}{D}} \leqq \frac{Cz}{D}$, we have $\alpha_n \leqq (C+1)z$ ($n=0, \ldots, M$). On the other hand, by $\alpha_0 \beta_0 = C$, $0 < \beta_0 < \alpha_0$ we have $\alpha_0 > 1$. Hence $\varepsilon^n < (C+1)z$, whence $M \leqq \frac{\log (C+1)z}{\log \varepsilon_0} \leqq c_1 \log 2Cz$ follows. This completes the proof of Lemma 1.

Corollary. *For positive integers A, B, C let $N(z; A, B, C)$ denote the number of solutions of*

$$Ax^2 - By^2 = C \tag{2. 3}$$

in positive integers x, y, $x \leqq z$. Then

$$N(z; A, B, C) \leqq N(Az; AB, AC) \leqq c_1 d(A^2 C^2) \log 2A^2 Bz.$$

This is obvious. If (x, y) is a solution of (2. 3) then (Ax, y) is a solution of $X^2 - ABY^2 = AC$ which proves the Corollary.

Lemma 2. (Borel—Cantelli) *Let $A_1, A_2, \ldots$ be an infinite sequence of sets in (Ω, A, P) and let $\sum_{j=1} P(A_j) < \infty$. Then almost all ω in Ω are belonging to finitely many A_i only.*

Proof of Theorem 1. Let $0 < i_1 < i_2 < \cdots < i_k$ be arbitrary but fixed integers. For a general integer n let $\bar{n} = (n+i_1)\ldots(n+i_k)$. Let us introduce the notation

$$\eta_N(\omega) = \sum_{n=1}^{N} f(\bar{n}, \omega); \quad M_{l,N} = \int_\Omega (\eta_N(\omega))^l \, dP. \tag{2. 4)-(2. 5}$$

First we give a non-trivial estimation for $M_{4,N}$, whence by using the Borel—Cantelli lemma we deduce that $\lim_{N\to\infty} \eta_N(\omega)/N = 0$ for almost all $\omega \in \Omega$.

It is obvious, that

$$M_{4,N} = \sum_{n_1, n_2, n_3, n_4} \int_\Omega f(\bar{n}_1 \bar{n}_2 \bar{n}_3 \bar{n}_4; \omega) \, dP,$$

where in the sum n_1, n_2, n_3, n_4 run independently over the values $1, 2, \ldots, N$. Using $\int_\Omega f(m; \omega) dP = 1$ or 0 according to m is a square-number, or not, we have that $M_{4,N}$ is equal to the number of solutions of the equation

$$\bar{n}_1 \bar{n}_2 \bar{n}_3 \bar{n}_4 = X^2 \tag{2. 6}$$

in unknowns n_1, n_2, n_3, n_4, X, satisfying $1 \leqq n_i \leqq N$ $(i = 1, 2, 3, 4)$.

For a fixed square-free integer $E(>0)$ let $H(E)$ denote the number of solutions of the equation

$$\bar{n}_1\bar{n}_2 = EY^2; \quad 1 \leqq n_1 \leqq N, \quad 1 \leqq n_2 \leqq N$$

in unknowns n_1, n_2, Y.

It is obvious that if n_1, n_2, n_3, n_4 is a solution of (2. 6) then the square-free parts of the numbers $\bar{n}_1\bar{n}_2$, $\bar{n}_3\bar{n}_4$ are the same. Hence we have

$$M_{4,N} = \sum_E H^2(E),$$

and consequently

$$M_{4,N} \leqq \left(\max_E H(E)\right) \sum_E H(E). \tag{2.7}$$

Observing that $\sum_E H(E) = N^2$ (since the number of the choice of all pairs n_1, n_2, $1 \leqq n_i \leqq N$ is N^2) we have

$$M_{4,N} \leqq N^2 \max_E H(E). \tag{2.8}$$

Now we estimate $H(E)$. For a general positive square-free A let $G(A)$ denote the number of $n \leqq N$ which can be written in the form

$$\bar{n} = AZ^2, \tag{2.9}$$

where Z is a suitable integer. Then we have

$$H(E) \leqq \sum_{E_1E_2=E} \sum_U G(E_1U)G(E_2U), \tag{2.10}$$

where in the right hand side E_1 runs over the divisors of E and U over the set of all square-free integers coprime to E.

For $k=1$ we evidently have $G(A) \leqq \sqrt{N/A}$. Consequently by (2. 10)

$$H(E) \leqq \sum_{E_1E_2=E} \frac{N}{\sqrt{E}} \cdot \sum_{U \leqq N} \frac{1}{U} \leqq \frac{N \log N}{\sqrt{E}} d(E) \leqq cN \log N,$$

and hence by (2. 8)

$$M_{4,N} \leqq cN^3 \log N. \tag{2.11}$$

Assume now that $k \geqq 2$. Consider the solutions of $\bar{n} = AZ^2$. Since the numbers $n+i_{j_1}$, $n+i_{j_2}$ have no common prime-divisors greater than $i_{j_2} - i_{j_1}$ if $j_1 \neq j_2$, for an n satisfying (2. 9) we have

$$n + i_j = R_jC_jZ_j^2 \qquad (j = 1, 2, \ldots, k), \tag{2.12}$$

where R_j, C_j are square-free numbers, the prime factors of R_j are not greater than

$i_k - i_1$ and the prime factors of C_j are greater than $i_k - i_1$ and $\prod_{j=1}^{k} C_j | A$. If n is a solution of (2. 12), then

(2.13) $$i_2 - i_1 = R_2 C_2 Z_2^2 - R_1 C_1 Z_1^2$$

holds with suitable $Z_1, Z_2 \leqq N$. Using the Corollary to Lemma 1 we have that the number of solutions of (2. 13) with $Z_1, Z_2 \leqq N$ is at most $c_1 d((R_1 C_1 (i_2 - i_1))^2) \log N \leqq \leqq c_1 N^{\varepsilon_1}$.

The number of all possible pairs of R_1, R_2 occurring in (2. 12) is bounded for fixed $i_1, i_2, \dots, i_k$. The number of couples (R_1, R_2) is at most $d^2(A) \leqq cN^{\varepsilon_2}$, since $C_1 C_2 | A$. Therefore

(2.14) $$G(A) \leqq cN^{\varepsilon}.$$

Using (2. 10) and the fact that the number of those A which occur as the square-free part of a number $\bar{n}$ for some $n \leqq N$ is at most N, we have

$$H(E) \leqq cN^{1+\varepsilon}.$$

Hence by (2. 8)

(2.15) $$M_{4,N} \leqq cN^{3+\varepsilon}$$

follows.

Using (2. 11) or (2. 15) according as $k=1$ or $k \geqq 2$, we have

(2.16) $$P(|\eta_N| > N^{\alpha}) \leqq \int_{\Omega} \frac{|\eta_N|^4}{N^{4\alpha}} dP < cN^{3-4\alpha+\varepsilon}.$$

Let $N_m = m^5$ and $\alpha = \frac{4}{5} + \varepsilon$. By (2. 16) we have

$$\sum_{m=1}^{\infty} P\left(|\eta_{N_m}| > N_m^{\frac{4}{5}+\varepsilon}\right) \leqq c \sum_{m=1}^{\infty} m^{-1-\varepsilon} < \infty.$$

Consequently by Lemma 2 we have

(2.17) $$\lim_{m \to \infty} \frac{\eta_{N_m}(\omega)}{N_m^{\frac{4}{5}+2\varepsilon}} = 0$$

for all fixed $\varepsilon > 0$ and for almost all $\omega \in \Omega$. Since for $N_m \leqq N < N_{m+1}$

(2.18) $$|\eta_N - \eta_{N_m}| \leqq N - N_m \leqq N_{m+1} - N_m \leqq cm^4 \leqq cN_m^{\frac{4}{5}} < cN^{\frac{4}{5}},$$

therefore by (2. 17)

$$\lim_{N \to \infty} \frac{\eta_N(\omega)}{N^{\frac{4}{5}+2\varepsilon}} = 0$$

for all $\varepsilon > 0$ and almost all $\omega \in \Omega$.

Finally we remark, that a function $f(n)\in\mathcal{F}$ is of normal type if and only if $\lim\limits_{N\to\infty}\frac{1}{N}\sum\limits_{n=1}^{N} f(\bar{n})=0$ for all choice of $k=1, 2, \dots$ and of $(i_1, \dots, i_k)$. This completes the proof of the theorem.

3. Theorem 2. *Let $f(n)$ be a completely multiplicative function, all values of which are $+1$ or -1. Assume that there exist at least two primes p_1, p_2 for which $f(p_1)=f(p_2)=-1$. Then for arbitrary ε_1, ε_2 $(\varepsilon_1=\pm 1,\ \varepsilon_2=\pm 1)$ there exist infinitely many n satisfying $f(n)=\varepsilon_1$, $f(n+1)=\varepsilon_2$.*

For $(\varepsilon_1, \varepsilon_2)=(+1, +1)$ or $(-1, 1)$ we can prove a stronger assertion. This is stated in Theorems 3 and 4.

Theorem 3. *Assuming that the series*

$$\sum_{f(p)=-1}\frac{1}{p} \tag{3.1}$$

diverges we have

$$\liminf_{x\to\infty} x^{-1}N_f(x, 1, 1)\geqq\frac{1}{12}, \tag{3.2}$$

$$\liminf_{x\to\infty} x^{-1}N_f(x, -1, -1)\geqq\frac{1}{12}, \tag{3.3}$$

where $N_f(x, \varepsilon_1, \varepsilon_2)$ denotes the number of those n not exceeding x for which $f(n)=\varepsilon_1$, $f(n+1)=\varepsilon_2$. Consequently

$$\liminf_{x\to\infty}\frac{1}{x}\sum_{n\leqq x} f(n)f(n+1)\geqq-\frac{5}{6}. \tag{3.4}$$

Let $\mathcal{P}$ be the set of those primes p for which $f(p)=-1$.

Theorem 4. *Suppose that $\mathcal{P}$ contains at least two elements and that the series $\sum\limits_{p\in\mathcal{P}}\frac{1}{p}$ converges. Then for both values $\varepsilon=1, -1$ we have*

$$\lim_{x\to\infty} x^{-1}N_f(x, \varepsilon, \varepsilon)=\frac{1}{4}\left(1+2\varepsilon\prod_{p\in\mathcal{P}}\frac{p-1}{p+1}+\prod_{p\in\mathcal{P}}\frac{p-3}{p+1}\right)\left(\overset{\text{def}}{=}A\right). \tag{3.5}$$

The number standing on the right-hand side of (3.5) *is positive.*

Proof of Theorems 2, 3, and 4. First we prove Theorem 2 for $(\varepsilon_1, \varepsilon_2)=(1, -1)$ and $(-1, 1)$. The remaining two cases will follow from Theorems 3 and 4.

The assertion of Theorem 2 for $(\varepsilon_1, \varepsilon_2)=(1, -1)$ and $(-1, 1)$ is equivalent to saying that $f(n)$ assumes both of the values $+1$ and -1 infinitely many times. For

$+1$ this is true since $f(n^2) = +1$ for all n. To show this for -1 let p be a prime for which $f(p) = -1$. Then $f(p^{2k+1}) = -1$ for all k.

To prove Theorem 3 we need a theorem due to E. WIRSING [1], which we state as

LEMMA 3. *If* $f(n) = \pm 1$ *and the series* (3. 1) *diverges, then*

(3. 6) $$x^{-1} \sum_{x \leqq x} f(n) \to 0 \quad as \quad x \to \infty.$$

Let $n_1 < n_2 < \cdots < n_L \leqq x$ be the sequence of those integers for which $f(n_i) = -1$. Let $m_1 < m_2 < \cdots < m_R \leqq x$ denote the complementary sequence, i.e. for which $f(m_i) = +1$. Let $\varrho_k(x)$ denote the number of those n_i for which $n_{i+1} - n_i = k$, $n_i \leqq x$. Similarly, let $\tau_k(x)$ denote the number of m's satisfying $m_{i+1} - m_i = k$, $m_i \leqq x$. From (3. 6) we easily deduce

(3. 7) $$L + O(1) = \sum_{k=1}^{\infty} \varrho_k(x) = \frac{x}{2} + o(x); \quad R + O(1) = \sum_{k=1}^{\infty} \tau_k(x) = \frac{x}{2} + o(x)$$

(3. 8) $$\sum_{k=1}^{\infty} k\varrho_k(x) = x + o(x), \quad \sum_{k=1}^{\infty} k\tau_k(x) = x + o(x).$$

Hence

(3. 9) $$\sum_{k=3}^{\infty} (k-2)\varrho_k(x) = \varrho_1(x) + o(x), \quad \sum_{k=3}^{\infty} (k-2)\tau_k(x) = \tau_1(x) + o(x)$$

follow. Consequently

(3. 10) $$\sum_{k \neq 2} k\varrho_k(x) \leqq 4\varrho_1(x) + o(x), \quad \sum_{k \neq 2} k\tau_k(x) \leqq 4\tau_1(x) + o(x),$$

(3. 11) $$\sum_{k \neq 2} \varrho_k(x) \leqq 2\varrho_1(x) + o(x), \quad \sum_{k \neq 2} \tau_k(x) \leqq 2\tau_1(x) + o(x).$$

Now we prove that $\liminf_{x \to \infty} x^{-1} \varrho_1(x) \geqq \frac{1}{12}$. The proof of the relation $\liminf_{x \to \infty} x^{-1} \tau_1(x) \geqq \frac{1}{12}$ is similar, and so we omit it. Since from (3. 6)

$$\frac{1}{x} \sum_{2n \leqq x} f(2n) \to 0$$

follows, among the n's the number of even numbers is $\frac{x}{4} + o(x)$. Hence by (3. 11) we have that there are at least $\frac{x}{4} - 2\varrho_1(x) - o(x)$ even n's satisfying $n_{i+1} - n_i = 2$. Let $\mathcal{S}$ denote the set of these n's.

We distinguish two cases.

Case a). $f(2)=1$. Then for $n_i \in \mathcal{S}$ the integers $n_i/2$ and $n_{i+1}/2$ are consecutive numbers, and furthermore $f\left(\frac{n_i}{2}\right)=f\left(\frac{n_{i+1}}{2}\right)=-1$, $\frac{n_i}{2} \leqq \frac{x}{2}$. Thus we have

$$\varrho_1\left(\frac{x}{2}\right) \geqq \frac{x}{4}-2\varrho_1(x)-o(x),$$

whence $3\varrho_1(x) \geqq \frac{x}{4}-o(x)$, i.e. $\liminf \frac{\varrho_1(x)}{x} \geqq \frac{1}{12}$, follows.

Case b). $f(2)=-1$. Then, for $n_i \in \mathcal{S}$, $\frac{n_i}{2}$ and $\frac{n_{i+1}}{2}$ are consecutive integers, and moreover $f\left(\frac{n_i}{2}\right)=f\left(\frac{n_{i+1}}{2}\right)=+1$, $\frac{n_i}{2} \leqq \frac{x}{2}$. Consequently

$$\tau_1\left(\frac{x}{2}\right) \geqq \frac{x}{4}-2\varrho_1(x)+o(x). \tag{3. 12}$$

Since the interval $[m_i, m_{i+1}]$ for $m_{i+1}-m_i = k$, $k \geqq 3$ contains $(k-1)$ elements from among the n's, we deduce that

$$\varrho_1(x) \geqq \sum_{k=3}^{\infty} (k-2)\tau_k(x);$$

hence by (3. 12)

$$\varrho_1(x) \geqq \tau_1(x)+o(x) \tag{3. 13}$$

follows. From here by (3. 12) we obtain

$$3\varrho_1(x) \geqq \frac{x}{4}+o(x),$$

i.e.

$$\lim_{x\to\infty} x^{-1}\varrho_1(x) \geqq \frac{1}{12}.$$

Now we prove Theorem 4. For this we need the following

Lemma 4. [2] *If $h(n)$ is a complex-valued completely multiplicative function satisfying the conditions: a) $|h(n)| \leqq 1$ $(n=1,2,\ldots)$, and b) $\sum_p \frac{h(p)-1}{p}$ converges, then*

$$\lim_{x\to\infty} x^{-1}\sum_{n\leqq x} h(n)h(n+1) = \prod_p \left(1+2\sum_{\alpha=1}^{\infty}\frac{h(p^\alpha)-h(p^{\alpha-1})}{p^\alpha}\right),$$

$$\lim_{x\to\infty} x^{-1}\sum_{n\leqq x} h(n) = \prod_p \left(1+\sum_{\alpha=1}^{\infty}\frac{h(p^\alpha)-h(p^{\alpha-1})}{p^\alpha}\right).$$

Observing that the conditions of Lemma 4 are satisfied for $h(n)=f(n)$ and that

$$4N(x, \varepsilon, \varepsilon) = \sum_{n\leqq x} (f(n)+\varepsilon)(f(n+1)+\varepsilon) = \sum_{n\leqq x} f(n)f(n+1)+2\varepsilon \sum_{n\leqq x} f(n)+x+0(1),$$

by Lemma 4 we obtain (3. 5).

Finally we prove the positivity of A. If $3\in\mathcal{P}$, then

$$A \geqq \frac{1}{4}\left(1-2\cdot\frac{2}{4}\prod_{p\in\mathcal{P},\, p\neq 3}\frac{p-1}{p+1}\right).$$

Since $\prod_{\substack{p\in\mathcal{P}\\ p\neq 3}}$ is not an empty product, it must be smaller than 1; so indeed $A>0$. If $3\notin P$, $2\in\mathcal{P}$, then

$$A \geqq \frac{1}{4}\left(1-\frac{2}{3}\prod_{p\in\mathcal{P},\, p>3}\frac{p-1}{p+1}-\frac{1}{3}\prod_{p\in\mathcal{P},\, p>3}\frac{p-3}{p+1}\right).$$

Using the fact that the products on the right hand side are not empty, we again have $A>0$. If 2, 3 are not belonging to $\mathcal{P}$, then

$$A \geqq \frac{1}{4}\left(1-2\prod_{p\in\mathcal{P},\, p>3}\frac{p-1}{p+1}+\prod_{p>3}\frac{p-3}{p+1}\right).$$

Using the relation $\frac{p-3}{p+1} < \left(\frac{p-1}{p+1}\right)^2$ for $p\geqq 3$,

$$A \geqq \frac{1}{4}\left(1-\prod_{p\in\mathcal{P}}\frac{p-1}{p+1}\right)^2 > 0$$

also in this case.

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Products of contractions in Hilbert space

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The aim of this Note is to study the convergence of some infinite products of contractions acting on a Hilbert space. We extend the results of I. HALPERIN ([4], Th. 1) and F. BROWDER ([1], Lemma 3) concerning products of projections to a larger class of operators.

Throughout this Note, H will denote a Hilbert space, $\mathcal{S}(H)$ the unit ball of the algebra of all bounded linear operators acting on H, and I the identity operator on H. The set of all natural numbers is denoted by N and if $T_j \in \mathcal{S}(H)$ $(j \in N)$ we set $T_n T_{n-1} \cdots T_1 = \prod_{j=1}^{n} T_j$. We also set for $\varepsilon \geqq 0$ and $T \in \mathcal{S}(H)$:

$$\varphi_T(\varepsilon) = \sup_{\substack{\|x\| \leqq 1 \\ \|x\| - \|Tx\| \leqq \varepsilon}} \|x - Tx\|, \quad \text{and} \quad \mathcal{S}_\varphi(H) = \{T \in \mathcal{S}(H) : \lim_{\varepsilon \to 0} \varphi_T(\varepsilon) = 0\}.$$

For any $T \in \mathcal{S}(H)$ the function φ_T is increasing and if $\|x\| \leqq 1$ we have

$$\|x - Tx\| \leqq \sup \{\|y - Ty\| : \|y\| \leqq 1, \ \|y\| - \|Ty\| \leqq \|x\| - \|Tx\|\} = \varphi_T(\|x\| - \|Tx\|).$$

Definition. A map ψ defined on N and valued in an arbitrary set is called *permissible* if for any $k \in N$ there is an $r_k \in N$ such that $\psi(k)$ belongs to the image by ψ of each block of r_k successive natural numbers (see [1], Def. 6).

Lemma 1. *Let* $T, S \in \mathcal{S}_\varphi(H)$. *We have* $TS \in \mathcal{S}_\varphi(H)$, $\operatorname{Ker}(I - TS) = \operatorname{Ker}(I - T) \cap \operatorname{Ker}(I - S)$.

Proof. For any $x \in H$ such that $\|x\| \leqq 1$, $\|x - TSx\| \leqq \varepsilon$ we have

$$\|x - TSx\| \leqq \|x - Tx\| + \|Tx - TSx\| \leqq \|x - Tx\| + \|x - Sx\| \leqq \varphi_T(\|x\| - \|Tx\|) + \\ + \varphi_S(\|x\| - \|Sx\|).$$

Using the inequalities $\|TSx\| \leqq \|Sx\|$, $\|TSx\| \leqq \|T(Sx - x)\| + \|Tx\|$ we also obtain

$$\|x\| - \|Sx\| \leqq \|x\| - \|TSx\| \leqq \varepsilon,$$

$$\|x\| - \|Tx\| \leqq \|T(x - Sx)\| + \|x\| - \|TSx\| \leqq \varphi_S(\varepsilon) + \varepsilon.$$

It results $\varphi_{TS}(\varepsilon) \leqq \varphi_T(\varphi_S(\varepsilon)+\varepsilon)+\varphi_S(\varepsilon)$, which implies $\lim \varphi_{TS}(\varepsilon)=0$ as $\varepsilon \to 0$, and $TS \in \mathscr{S}_\varphi(H)$.

Now if $x=TSx$, $\|x\| \leqq 1$ we have $\|x\|=\|Sx\|$, $\|x-Sx\| = \lim_{\varepsilon \to 0} \varphi_S(\varepsilon) = 0$, thus $x=Sx=Tx$, $\operatorname{Ker}(I-TS) \subset \operatorname{Ker}(I-T) \cap \operatorname{Ker}(I-S)$. The opposite inclusion is evident.

Lemma 2. *Let $\mathscr{A} \subset \mathscr{S}_\varphi(H)$ such that $\lim_{\varepsilon \to 0} \sup_{T \in \mathscr{A}} \varphi_T(\varepsilon)=0$. Then for any sequence $\{T_j\}_{j \in N}$ we have $\lim_{n \to \infty} \|(\prod_{j=1}^{n} T_j - \prod_{j=1}^{n+1} T_j)x\| = 0$.*

Proof. Let $x \in H$, $\|x\| \leqq 1$. Since $\{\|\prod_{j=1}^{n} T_j x\|\}_{n \in N}$ is a convergent sequence one obtains

$$\lim_{n \to \infty} \|(\prod_{j=1}^{n} T_j - \prod_{j=1}^{n+1} T_j)x\| \leqq \lim_{n \to \infty} \varphi_T(\|\prod_{j=1}^{n} T_j x\| - \|\prod_{j=1}^{n+1} T_j x\|) \leqq$$

$$\leqq \lim_{n \to \infty} \sup_{T \in \mathscr{A}} \varphi_T(\|\prod_{j=1}^{n} T_j x\| - \|\prod_{j=1}^{n+1} T_j x\|) = 0.$$

Theorem 1. *Let $\mathscr{A}$ be a commutative subset of $\mathscr{S}_\varphi(H)$ and $\psi: N \to \mathscr{A}$ such that $\psi^{-1}(\psi(k))$ is an infinite set for any $k \in N$. Then $\{\prod_{j=1}^{n} T_j\}_{n \in N}$ converges strongly to the orthogonal projection P of H onto $\bigcap_{k=1}^{\infty} \operatorname{Ker}(I-\psi(k))$.*

Proof. For simplicity we introduce the notation $T_n = \prod_{j=1}^{n} \psi(j)$, $N_k=\psi^{-1}(\psi(k))$. We have $\langle T_{n+1}^* T_{n+1} x, x\rangle = \langle T_n^* \psi(n+1)^* \psi(n+1) T_n x, x\rangle \leqq \langle T_n^* T_n x, x\rangle$, thus by [6], Sec. 104 (p. 261) the sequence $\{T_n^* T_n\}_{n \in N}$ converges strongly to a positive operator A.

Let $y \in H$, $\|y\| \leqq 1$. We have

$$\lim_{n+1 \in N_k} \|(I-\psi(k)T_n y\| \leqq \lim_{n+1 \in N_k} \varphi_{\psi(k)}(\|T_n y\| - \|\psi(k) T_n y\|) =$$

$$= \lim_{n+1 \in N_k} \varphi_{\psi(k)}(\|T_n y\| - \|T_{n+1} y\|) = 0,$$

consequently

$$\langle (I-\psi(k)^*)Ax, y\rangle = \lim_{n+1 \in N_k} \langle (I-\psi(k)^*) T_n^* T_n x, y\rangle =$$

$$= \lim_{n+1 \in N_k} \langle T_n x, (I-\psi(k)) T_n y\rangle = 0.$$

Because the kernels of $I-\psi(k)$ and $I-\psi(k)^*$ coincide (see [5], Sec. I. 3. 1) we infer

$$(I-\psi(k)A = (I-\psi(k)^*)A = 0, \; T_n^* T_n A = A$$

which shows that A is a projection and $A \leqq P$. But we have also $\langle Px, x\rangle = \langle T_n^* P T_n x, x\rangle \leqq \langle T_n^* T_n x, x\rangle$ thus $P \leqq A$. It follows $P = A$ and

$$\lim_{n\to\infty} \|T_n x - Px\|^2 = \lim_{n\to\infty} \langle T_n^* T_n x - Px, x\rangle = \langle Ax - Px, x\rangle = 0$$

which concludes the proof.

Corollary 1. *Let* $T_j \in \mathscr{S}_\varphi(H)$, $j = 1, 2, \ldots, m$ *and put* $T = T_1, T_2, \ldots, T_m$. *Then* $\{T^n\}_{n\in N}$ *converges strongly to the orthogonal projection of* H *onto* $\bigcap_{j=1}^{m} \operatorname{Ker}(I - T_j)$.

Proof. By lemma 1 we have $T \in \mathscr{S}_\varphi(H)$, $\operatorname{Ker}(I-T) = \bigcap_{j=1}^{m} \operatorname{Ker}(I - T_j)$. Set $\mathscr{A} = \{T\}$, $\psi(j) = T$. We have $T^n = \prod_{j=1}^{n} \psi(j)$ thus we can apply Th. 1.

Theorem 2. *Let* $\mathscr{A} \in \mathscr{S}_\varphi(H)$ *such that* $\lim_{\varepsilon\to 0} \sup_{T\in\mathscr{A}} \varphi_T(\varepsilon) = 0$ *and let* $\psi: N \to \mathscr{A}$ *be a permissible map. Then* $\{\prod_{j=1}^{n} \psi(j)\}_{n\in N}$ *converges weakly to the orthogonal projection of* H *onto* $\bigcap_{j=1}^{\infty} \operatorname{Ker}(I - \psi(j))$.

Proof. Let P be the orthogonal projection of H onto $\bigcap_{j=1}^{\infty} \operatorname{Ker}(I - \psi(j))$. We have $\psi(j)P = P$, $\psi(j)^* P = P$, thus $\prod_{j=1}^{n} \psi(j) P = P \prod_{j=1}^{n} \psi(j) = P$ and

$$\lim_{n\to\infty} \langle (\prod_{j=1}^{n} \psi(j) - P)x, y\rangle = \lim_{n\to\infty} \langle (I-P) \prod_{j=1}^{n} \psi(j) x, y\rangle.$$

Suppose the subsequence $\{\prod_{j=1}^{m_n} \psi(j)x\}_{n\in N}$ converges weakly to z and take $k \in N$. By the definition of ψ there is $r_k \in N$ and $s_n \in N$ such that $\psi(s_n) = \psi(k)$, $r_k \geqq m_n - s_n \geqq 0$. Using Lemma 2 we get

$$z = \lim_{n\to\infty} \prod_{j=1}^{m_n} \psi(j)x = \lim_{n\to\infty} \prod_{j=1}^{s_n - 1} \psi(j)x = \lim_{n\to\infty} \prod_{j=1}^{s} \psi(j)x = \psi(k)z;$$

consequently $Pz = z$.

Now if $\{\prod_{j=1}^{n} \psi(j)\}$ does not converge weakly to P we can find $x, y \in H$ and a subsequence $\{m_n\}_{n\in N}$ such that $\lim_{n\to\infty} \langle (\prod_{j=1}^{m_n} \psi(j) - P)x, y\rangle = a \neq 0$.

But we may suppose that $\{\prod_{j=1}^{m_n} \psi(j)x\}_{n\in N}$ converges weakly to $z = Pz$ because balls in H are weakly compact and we get a contradiction:

$$0 \neq a = \lim_{n\to\infty} \langle (I-P) \prod_{j=1}^{m_n} \psi(j)x, y\rangle = \langle (I-P)z, y\rangle = 0.$$

Remark. The set of all orthogonal projections in H is contained in $\mathcal{S}_\varphi(H)$. If $P(=P^*)$ is a projection we have $\varphi_P(\varepsilon)\leqq\sqrt{2\varepsilon}$ thus Cor. 1 and Th. 1. are equally applicable to projection operators (results of I. HALPERIN and F. BROWDER). Let σ be a closed set in the complex plane included in the unit disc D and such that $\sigma\cap\{\lambda:|\lambda|=1\}$ contains at most the point 1. If T is a normal operator, $\sigma(T)\subset\sigma$, with the spectral measure $E(\cdot)$, let us put $\sigma_\varepsilon=\left\{\lambda\in\sigma:|\lambda|\leqq\sqrt{1-\sqrt{\varepsilon}}\right\}$, $E(\sigma_\varepsilon)=P_\varepsilon$, $I-E(\sigma_\varepsilon)=Q_\varepsilon$. For any $x\in H$, $\|x\|\leqq 1$, $\|x\|-\|Tx\|\leqq\varepsilon$ we have

$$2\varepsilon\geqq\|x\|^2-\|Tx\|^2=\|P_\varepsilon x\|^2-\|TP_\varepsilon x\|^2+\|Q_\varepsilon x\|^2-\|TQ_\varepsilon x\|^2\geqq\|P_\varepsilon x\|^2-$$

$$-\|TP_\varepsilon x\|^2\geqq\sqrt{\varepsilon}\,\|P_\varepsilon x\|^2$$

thus $\|P_\varepsilon x\|\leqq\sqrt{2\sqrt{\varepsilon}}$ and $\|x-Tx\|\leqq 2\sqrt{2\sqrt{\varepsilon}}+\|Q_\varepsilon x-TQ_\varepsilon x\|\leqq 2\sqrt{2\sqrt{\varepsilon}}+\|(T|Q_\varepsilon H)\|$. It follows $\varphi_T(\varepsilon)\leqq\sqrt{2\sqrt{\varepsilon}}+\sup\limits_{\lambda\in\sigma-\sigma_\varepsilon}|1-\lambda|$ thus $T\in\mathcal{S}_\varphi(H)$. In fact we showed that Th. 2 is applicable to the set $\mathcal{A}=\{T\in\mathcal{S}(H);\ T \text{ normal},\ \sigma(T)\subset\sigma\}$.

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On a certain class of representations of function algebras

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1. Introduction. In [4] B. SZ.-NAGY and C. FOIAŞ have introduced the class $\mathscr{C}_\varrho$ of all bounded linear operators T on the Hilbert space H, which admit a representation of the form:

(1) $$T^n = \varrho P_H U^n | H \qquad (n=1, 2, \ldots),$$

where U is a unitary operator on a Hilbert space K, containing H as a subspace and P_H is the orthogonal projection of K onto H. In [3] they have proved that any T belonging to some class $\mathscr{C}_\varrho$ is similar to a contraction.

The definition of the class $\mathscr{C}_\varrho$ has a natural correspondent for operator valued representations on Hilbert spaces. For this let X be a compact Hausdorff space, $C(X)$ the Banach algebra of all complex-valued continuous functions on X, A a function algebra on X (i. e. a closed subalgebra of X, which contains the constants and separates the points of X), and M_A the maximal ideal space of A (i.e. the set of all complex homomorphisms of A). For any $\Phi \in M_A$ there exists a positive measure m on X such that

$$\Phi(f) = \int f\, dm \qquad (f \in A).$$

Such a measure is called a representing measure for Φ (see [6]). As usual we write A_Φ for the kernel of Φ.

By a *representation* of A on H we shall mean an algebraic homomorphism $f \to T_f$ of A in $\mathscr{B}(H)$ (the algebra of all bounded linear operators on H) satisfying $T_1 = I$ (the identical operator on H) and

$$\|T_f\| \leqq k \|f\| \qquad (f \in A).$$

If $k=1$, $f \to T_f$ is called a *contractive representation* of A on H.

Let $\varrho > 0$. A (contractive) representation $\varphi \to U_\varphi$ of $C(X)$ on a Hilbert space K, where K contains H as a subspace, *will be called a spectral ϱ-dilation of $f \to T_f$ with respect to $\Phi \in M_A$, if*

(2) $$T_f = \varrho P_H U_f | H \qquad (f \in A_\Phi).$$

We say that a representation of A on H is of *class* $\mathscr{C}_\varrho(A, H)$ if it has a spectral ϱ-dilation. If $\varrho=1$, the spectral ϱ-dilation of $f\to T_f$ means simply the *spectral dilation* of $f\to T_f$ (see [2]). A contractive representation for which there exists a spectral dilation is called a *dilatable representation.*

The purpose of this note is to prove the analog of the result in [3], in the context of representations of function algebras. This is contained in the following

Theorem. *Let $f\to T_f$ be a representation of class $\mathscr{C}_\varrho(A, H)$ with respect to $\Phi\in M_A$. Then there exists a Hilbert space H', an affinity X of H' onto H, and a contractive representation $f\to T'_f$ of A on H' such that*

$$T_f X = X T'_f \qquad (f\in A).$$

Moreover, $f\to T'_f$ is a dilatable representation, and the spectral ϱ-dilation of $f\to T_f$ is a spectral dilation of $f\to T'_f$.

2. Firstly we get a caracterization of the classes $\mathscr{C}_\varrho(A, H)$ and the monotonity of these classes. For this aim let $f\to T_f$ be a representation of class $\mathscr{C}_\varrho(A, H)$ and $\varphi\to U_\varphi$ its spectral ϱ-dilation. If $f\in A$, relation (2) implies:

$$\varrho P_H U_f|H = \varrho P_H U_{f-\Phi(f)}|H+\varrho\Phi(f)I = T_f+(\varrho-1)\Phi(f)I,$$

that is,

(3)
$$\frac{1}{\varrho}T_f+\left(1-\frac{1}{\varrho}\right)\Phi(f)I = P_H U_f|H \qquad (f\in A).$$

Now $\varphi\to S_\varphi=P_H U_\varphi|H$ $(\varphi\in C(X))$ is a positive map of $C(X)$ into $\mathscr{B}(H)$ (see [1]) for which the spectral dilation is exactly $\varphi\to U_\varphi$. Now T_f has the form:

$$T_f = \varrho S_f+(1-\varrho)\Phi(f)I = \varrho S_f+(1-\varrho)\left(\int f\,dm\right)I,$$

where m is a fixed representing measure for Φ.

If we put

$$\tilde{T}_\varphi = \varrho S_\varphi+(1-\varrho)\left(\int \varphi\,dm\right)I \qquad (\varphi\in C(X))$$

we obtain a linear map $\varphi\to\tilde{T}_\varphi$ of $C(X)$ into $\mathscr{B}(H)$, which extends the given representation and satisfies

$$\frac{1}{\varrho}\tilde{T}_\varphi+\left(1-\frac{1}{\varrho}\right)\left(\int\varphi\,dm\right)I\geqq 0 \qquad (\varphi\geqq 0,\ \varphi\in C(X)).$$

The last condition is equivalent to

(4)
$$(\varrho-1)\left(\int\varphi\,dm\right)I+\tilde{T}_\varphi\geqq 0 \qquad (\varphi\geqq 0,\ \varphi\in C(X)).$$

Conversely if we are given a representation $f \to T_f$ of A on H, which admits an extension $\varphi \to \tilde{T}_\varphi$ to $C(X)$ satisfying (4), then

$$S_\varphi = \frac{1}{\varrho}\tilde{T}_\varphi + \left(1-\frac{1}{\varrho}\right)\left(\int \varphi\, dm\right) I$$

defines a positive map $\varphi \to S_\varphi$ of $C(X)$ into $\mathscr{B}(H)$. Let $\varphi \to U_\varphi$ be the spectral dilation of $\varphi \to S_\varphi$ (see [1]). It is immediate that $\varphi \to U_\varphi$ is a spectral ϱ-dilation of $f \to T_f$, and consequently the given representation is of class $\mathscr{C}_\varrho(A, H)$. In this manner we have proved the following

Proposition. *The representation $f \to T_f$ of A on H is of the class $\mathscr{C}_\varrho(A, H)$ if and only if it admits a linear extension $\varphi \to \tilde{T}_\varphi$ to $C(X)$ satisfying* (4).

Corollary. *If $\varrho \leqq \varrho'$ then $\mathscr{C}_\varrho(A, H) \subseteqq \mathscr{C}_{\varrho'}(A, H)$.*

Proof. Let $f \to T_f$ be a representation of the class $\mathscr{C}_\varrho(A, H)$. Then, by the proposition, it has an extension $\varphi \to \tilde{T}_\varphi$ to $C(X)$ which satisfies (4). But if $\varphi \in C(X)$, $\varphi \geqq 0$, then for $\varrho' \geqq \varrho$ we have $(\varrho'-1)\ (\int \varphi\, dm) I + \tilde{T}_\varphi \geqq (\varrho-1)\ (\int \varphi\, dm) I + \tilde{T}_\varphi \geqq 0$, that is, condition (4) is satisfied, with ϱ' instead of ϱ. According to the above proposition, $f \to T_f$ is of the class $\mathscr{C}_{\varrho'}(A, H)$, and the corollary is proved.

3. Now we are able to prove the theorem. This proof is modelled on that in [3]. In the sequel m will be a fixed representing measure for Φ.

We suppose that $f \to T_f$ is of class $\mathscr{C}_r(A, H)$. Then, by the corollary, it is also of class $\mathscr{C}_\varrho(A, H)$ for $\varrho \geqq r$. Let $\varphi \to U_\varphi$ be the spectral ϱ-dilation of $f \to T_f$, and K_ϱ the ϱ-dilation space. We set

$$M_\varrho = \bigvee_{f \in A_\Phi,\, g \in A} U_g^*(U_f^* - T_f^*) H \tag{5}$$

and $t_\varrho = \|P_{M_\varrho}|H\|$, where P_{M_ϱ} is the orthogonal projection of K_ϱ on M_ϱ. It is obvious that $t_\varrho \leqq 1$. Moreover, t_ϱ is the smallest positive number for which the inequality

$$|(h, m_\varrho)| \leqq t_\varrho \|h\|\, \|m_\varrho\| \tag{6}$$

holds for any $h \in H$ and $m_\varrho \in M_\varrho$ of the form:

$$m_\varrho = \sum_{g,f} U_g^*(U_f^* - T_f^*) h_g^f, \tag{7}$$

where the family $\{h_g^f : g \in A,\ f \in A_\Phi\}$ has a finite number of elements.

Using (3) we obtain by a simple computation:

$$(h, m_\varrho) = \left(h, \sum_{g,f} (\delta - 1)\overline{\Phi(g)}\, T_f^* h_g^f\right),$$

where $\delta = \frac{1}{\varrho}$. Consequently, relation (6) is equivalent to

$$(8) \qquad (\delta-1)^2 \Big\| \sum_{g,f} \overline{\Phi(g)} T_f^* h_g^f \Big\|^2 \leqq t_\varrho^2 \|m_\varrho\|^2.$$

Now we compute the norm of m_ϱ:

$$\|m_\varrho\|^2 = \sum_{g,g'} \Big(U_{g'\bar{g}} \sum_f (U_f^* - T_f^*) h_g^f, \sum_{f'} (U_{f'}^* - T_{f'}^*) h_{g'}^{f'} \Big) =$$

$$= \sum_{g,g'} \Big[\sum_{f,f'} (U_{f'g'\bar{g}\bar{f}} h_g^f, h_{g'}^{f'}) - \sum_{f,f'} (T_f^* h_g^f, U_{g\bar{g}'\bar{f}'} h_{g'}^{f'}) -$$

$$- \sum_{f,f'} (U_{g'\bar{g}\bar{f}} h_g^f, T_{f'}^* h_{g'}^{f'}) + \sum_{f,f'} (U_{g'\bar{g}} T_f^* h_g^f, T_{f'}^* h_{g'}^{f'}) \Big] =$$

$$= \sum_{\substack{g,g'\\ f,f'}} (h_g^f, h_{g'}^{f'}) \int f' g' \bar{f} \bar{g}\, dm - 2\,\mathrm{Re} \sum_{\substack{g,g'\\ f,f'}} (T_f^* h_g^f, h_{g'}^{f'}) \int f' g' \bar{g}\, dm +$$

$$+ \sum_{g,g'} (T_f^* h_g^f, T_{f'}^* h_{g'}^{f'}) \int g' \bar{g}\, dm + \frac{1}{\varrho} \sum.$$

In this calculus we have used:

$$(U_\varphi h, h') = (h, h') \int \varphi\, dm + \frac{1}{\varrho} \Big[(\tilde{T}_\varphi h, h') - (h, h') \int \varphi\, dm \Big] \qquad (h, h' \in H;\ \varphi \in C(X))$$

and we have denoted by $\frac{1}{\varrho} \Sigma$ the term which contains $\frac{1}{\varrho}$ as a factor.

By introducing the scalar products under the integral and interchanging the sum with the integral it follows

$$\|m_\varrho\|^2 = \int \Big\{ \Big\| \sum_{g,f} \bar{f} \bar{g} h_g^f \Big\|^2 - 2\,\mathrm{Re} \Big(\sum_{g,f} \bar{g} T_f^* h_g^f, \sum_{g',f'} \bar{g}' f' h_{g'}^{f'} \Big) + \Big\| \sum_{g,f} \bar{g} T_f^* h_g^f \Big\|^2 \Big\} dm + \frac{1}{\varrho} \sum =$$

$$\int \Big\| \sum_{g,f} \bar{f} \bar{g} h_g^f - \sum_{g,f} \bar{g} T_f^* h_h^f \Big\|^2 dm + \frac{1}{\varrho} \sum.$$

Now writing $m_r \in M_r$ as in (7) we obtain

$$(9) \qquad \varrho \|m_\varrho\|^2 - r \|m_r\|^2 = (\varrho - r) \int \Big\| \sum_{g,f} \bar{f} \bar{g} h_g^f - \sum_{g,f} \bar{g} T_f^* h_g^f \Big\|^2 dm.$$

By (9) and by a simple evaluation of the integral of the vector-valued continuous functions we deduce

$$\varrho \|m_\varrho\|^2 \geqq r \|m_r\|^2 + (\varrho - r) \Big\| \int \sum_{g,f} (\bar{f} \bar{g} h_g^f - \bar{g} T_f^* h_g^f)\, dm \Big\|^2 =$$

$$= r \|m_r\|^2 + (\varrho - r) \Big\| \sum_{g,f} \Big(\int \bar{g}\, dm \Big) T_f^* h_g^f \Big\|^2.$$

For the last equality we have used

$$\int \bar{f}\bar{g}h_g^f\,dm = \left(\int \bar{f}\bar{g}\,dm\right)h_g^f = \overline{\Phi(f)}\,\overline{\Phi(g)}h_g^f = 0.$$

Because (8) remains true if $\varrho=r$, with 1 instead of t_r we have

$$\varrho\|m_\varrho\|^2 \geqq \left[r\left(\frac{1}{r}-1\right)^2+(\varrho-r)\right]\|\sum_{g,f}\overline{\Phi(g)}T_f^*h_g^f\|^2 =$$

$$= \left(\varrho-2+\frac{1}{r}\right)\|\sum_{g,f}\overline{\Phi(g)}T_f^*h_g^f\|^2.$$

Now by multiplying with $\left(\frac{1}{\varrho}-1\right)^2$, a simple computation shows that

$$\left(1-\frac{1}{\varrho}\right)^2\|\sum_{g,f}\overline{\Phi(g)}T_f^*h_g^f\|^2 \leqq \frac{\varrho-2+\frac{1}{\varrho}}{\varrho-2+\frac{1}{r}}\|m_\varrho\|^2.$$

Comparing this inequality with (8) we conclude that $t_\varrho<1$ for $\varrho>r$.

The rest of the proof proceeds exactly the same way as in [3], with the only remark that $k\in N_\varrho=K_\varrho\ominus M_\varrho$ $(\varrho>r)$ if and only if

$$T_f P_H U_g k = P_H U_{gf} k \qquad (g\in A,\ f\in A_\Phi).$$

The desired space in the theorem is $H'=P_{N_\varrho}H$, the affinity is $X=\overline{P_H|H'}$, and finally $T_f'=P_{H'}U_f|H'$ $(f\in A)$.

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A theorem on boundary spectra

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Dedicated in sorrow to the memory of David Topping

1. Let T be a bounded linear operator on a Hilbert space H. For a complex number ζ, let $E_\zeta[T]=\{x\in H: Tx=\zeta x\}$; thus $E_\zeta[T]\neq(0)$ if and only if ζ is an eigenalue of T. As well known, the relation

$$E_\zeta[T]=E_{\bar\zeta}[T^*] \tag{1}$$

holds for normal operators, but does not hold in general for non-normal operators; in fact, it is obvious that no point ζ of the residual spectrum of T can satisfy (1). It does hold if $\|T\|\leqq 1$ and $|\zeta|=1$, see [6] or [5; p. 8]. SCHREIBER [7] generalized this result as follows. Let $A_\zeta[T]=\{\{x_n\}: x_n\in H,\ \|x_n\|=1,\ \|Tx_n-\zeta x_n\|\to 0\ (n\to\infty)\}$. Then

$$A_\zeta[T]=A_{\bar\zeta}[T^*] \tag{2}$$

holds if $\|T\|\leqq 1$ and $|\zeta|=1$.

SZ.-NAGY and C. FOIAŞ [4] proved the following theorem.

Theorem A. *Let X be a spectral set for T, and let ζ be a boundary point of X. If there exists a sequence $\{D_n\}$ of open disks $D_n=\{\alpha: |\alpha-\alpha_n|<r_n\}$ contained in the complement of X such that*

$$\alpha_n\to\zeta,\quad r_n^{-1}|\alpha_n-\zeta|\to 1\quad \textit{as}\quad n\to\infty,$$

then (1) *is true for T and ζ.*

If T is a contraction, the unit disk $X=\{\lambda: |\lambda|\leqq 1\}$ is a spectral set for T, and this theorem is a generalization of the above result for contractions. In the light of this fact it is natural to seek a generalization of (2).

2. In the following, we refer to the condition on ζ in Theorem A as condition A_X. Then our result is the following:

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Theorem 1. *Let X be a closed set containing the spectrum $\sigma(T)$ of an operator T, and suppose that*

(3) $$\|(T-\alpha I)^{-1}\| \leqq 1/d(\alpha, X)$$

for $\alpha \notin X$, where $d(\alpha, X)$ is the distance from α to X. Then relation (2) (*and hence* (1)) *holds for any point $\zeta \in X$ satisfying condition* A_X.

If X is a spectral set for T, i.e. if

$$\|f(T)\| \leqq \sup\{|f(\lambda)| : \lambda \in X\}$$

for all rational functions $f(\lambda)$ without poles in X, then T obviously satisfies (3).

If T is a hyponormal operator, condition (3) is satisfied. For, in this case, $(T-\alpha I)^{-1}$ $(\alpha \notin X)$ is also hyponormal [8], and so $(T-\alpha I)^{-1}$ is normaloid for $\alpha \notin X$. Thus we have

$$\|(T-\alpha I)^{-1}\| = 1/\inf\{|\lambda-\alpha| : \lambda \in \sigma(T)\} \leqq 1/\inf\{|\lambda-\alpha| : \lambda \in X\} = 1/d(\alpha, X).$$

A recent result [3] of Putnam is essentially the same as Theorem A, and Theorem 1 is also a generalization of [3] to the case of approximate eigenvalues.

The following proof of Theorem 1 is along the same lines as the argument in [4].

Proof (Theorem 1). Let $\{x_n\}$ be a sequence of unit vectors such that $\|T^*x_n-\bar{\zeta}x_n\| \to 0$ $(n \to \infty)$. Since $\alpha_k \notin \sigma(T) \subset X$ for each k, $T_k = r_k(T-\alpha_k I)^{-1}$ exists as a bounded operator for each k and $\|T_k^* x_n-\zeta_k x_n\| \to 0$ $(n \to \infty)$, where $\zeta_k = r_k(\zeta-\alpha_k)^{-1}$ (see [9: Lemma 4]). By the growth condition (3)

$$\|T_k\| = r_k\|(T-\alpha_k I)^{-1}\| \leqq r_k/d(\alpha_k, X) \leqq 1,$$

and so T_k is a contraction. Let U_k be a unitary dilation of T_k. Then

$$\begin{aligned}\|(T_k-\zeta_k I)x_n\|^2 &= \|P(U_k-\zeta_k I)x_n\|^2 \leqq \|(U_k-\zeta_k I)x_n\|^2 = \|(U_k-\zeta_k I)^* x_n\|^2 = \\ &= 1-|\zeta_k|^2-2\,\mathrm{Re}\,\zeta_k((T_k-\zeta_k I)^* x_n, x_n) \leqq 1-|\zeta_k|^2+2|\zeta_k|\,\|(T_k-\zeta_k I)^* x_n\|.\end{aligned}$$

Since $T_k-\zeta_k I = r_k(T-\alpha_k I)^{-1}(\zeta I-T)(\zeta-\alpha_k)^{-1}$, we have

$$\begin{aligned}\|(T-\zeta I)x_n\| &\leqq r_k^{-1}|\zeta-\alpha_k|\,\|T-\alpha_k I\|\,\|(T_k-\zeta_k I)x_n\| \\ &\leqq r_k^{-1}|\zeta-\alpha_k|(\|T\|+|\alpha_k|)[1-|\zeta_k|^2+2|\zeta_k|\,\|(T_k-\zeta_k I)^* x_n\|]^{1/2} \leqq \\ &\leqq K\cdot[(r_k^{-1}|\zeta-\alpha_k|)^2-1+2r_k^{-1}|\zeta-\alpha_k|\,\|(T_k-\zeta_k I)^* x_n\|]^{1/2}\end{aligned}$$

for some constant $K>0$. (Note that $\{\alpha_k\}$ is bounded.) Let $\varepsilon>0$ be given. Then by condition A_X there exists a positive integer k such that

$$0 < (r_k^{-1}|\zeta-\alpha_k|)^2-1 < \left(\frac{\varepsilon}{2K}\right)^2.$$

For such a k (fixed), there exists an integer N such that

$$2r_k^{-1}|\zeta-\alpha_k|\,\|(T_k-\zeta_k I)^* x_n\| < \left(\frac{\varepsilon}{2K}\right)^2 \qquad (n > N),$$

because $\|(T_k-\zeta_k I)^* x_n\| \to 0$ $(n\to\infty)$ and $\{r_k^{-1}|\zeta-\alpha_k|\}$ is bounded. Thus we have

$$\|(T-\zeta I)x_n\| < \varepsilon \quad \text{for} \quad n > N.$$

$\varepsilon>0$ being arbitrary, $Tx_n-\zeta x_n \to 0$ $(n\to\infty)$. Hence we have $A_{\bar\zeta}[T^*] \subset A_\zeta[T]$. Since the argument is symmetric, relation (2) holds.

We note here that Theorem 1 covers the results of [9, § 3]. A point $\zeta\in X$ is called a semi-bare point [9] if there is a circle through ζ such that no points of X lie inside this circle. This notion is a generalization of bare point of a compact convex set in the plane. If $\zeta\in X$ is a semi-bare point, then ζ satisfies condition A_X. In fact, there exists an $\alpha_0\notin X$ such that $|\zeta-\alpha_0| = d(\alpha_0, X)$ by the definition of semi-bare point. Let $\{\alpha_n\}$ be a sequence of complex numbers lying on the line segment $\overline{\alpha_0\zeta}$ such that

$$\frac{1}{(n+1)^{1/2}} < |\alpha_n-\zeta| < \frac{1}{n^{1/2}}.$$

Then $D_n = \{\alpha : |\alpha-\alpha_n| < r_n\}$ with $r_n = |\alpha_n-\zeta| - \dfrac{1}{(n+1)^{1/2}}$ is contained in the complement of X for each n, and $\alpha_n\to\zeta$, $r_n^{-1}|\alpha_n-\zeta|\to 1$ as $n\to\infty$. By this consideration we have the following corollary [9: Theorems 2 and 3].

Corollary 2. *If T is a hyponormal operator, then the set of semi-bare points of the spectrum $\sigma(T)$ of T does not intersect the residual spectrum of T. In addition,* (2) *is satisfied for any semi-bare point ζ of $\sigma(T)$.*

Of course, Theorem 1 contains the result by Schreiber [7], and Corollary 1 in [3] can be generalized to the case of approximate point spectrum. We have (corresponding to Corollary 1 in [3]) the following

Corollary 3. *Let T satisfy* (3) *for $X=\sigma(T)$ and let the entire boundary of $\sigma(T)$ be a convex simple closed curve C. Then each $\zeta\in C$ satisfies* (2).

In fact, each point $\zeta\in C$ is a semi-bare point of $\sigma(T)$.

T. Yoshino kindly communicated the following result. It is published here with his permission.

Corollary 4. *Let T be a hyponormal operator such that $T = A+C$, where C is a compact operator and the spectrum of A lies on a straight line L. Then T is a normal operator.*

C. Putnam pointed out to the author (in a private communication) that this can be generalized to a more general result, but it is not in our context.

Proof. We can decompose T into $T = T_1 \oplus T_2$ where T_1 is normal and the point spectrum of T_2 is void [10, Lemma 1]. By [10, Lemma 2], each point of the continuous spectrum of T_2 belongs to the spectrum $\sigma(A)$ of A. By Corollary 2, every semi-bare point of the spectrum $\sigma(T)$ of T_2 is contained in the continuous spectrum of T_2 (note that the point spectrum of T_2 is void). Thus $\sigma(T_2)$ lies on L. Hence T_2 is normal, because a hyponormal operator with the spectrum on the real axis is self-adjoint (see [2, Theorem 1]).

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N^p-operators and semi-Carleman operators

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1. Let (X, μ) be a measure space, E a Banach space, and let p and p' be the usual conjugate numbers with $1 < p < +\infty$, that is $1/p + 1/p' = 1$. Let $L^p(X, \mu; E)$ be the Banach space of all equivalent classes of μ-strongly measurable E-valued functions K such that $\|K\|^p = \int_X \|K(x)\|^p d\mu < +\infty$.

Operators of the type $T: L^p(X, \mu) \to E$, which can be represented by a unique K in $L^{p'}(X, \mu; E)$ in the following way: $Tg = \int_X g(x)K(x)d\mu$ were considered by A. PERSSON. In [3] he showed that these are operators of type N^p which are also known as right p-nuclear operators. (See [1], Théorème 6.) The author proved in [7] that if E is the strong dual of some Banach space F such that either E is separable or reflexive, then T is the adjoint of an operator $S: F \to L^{p'}(X, \mu)$ such that $|Sf(x)| \leqq \gamma(x)\|f\|$ a.e. for some non-negative γ in $L^{p'}(X, \mu)$. In section 2 of this note we give a new characterization of this class of N^p-operators without referring to their adjoints. A necessary and sufficient condition for T to be of this class is that $\|Tg\| \leqq \int_X \gamma(x)|f(x)|d\mu$ for some non-negative γ in $L^{p'}(X, \mu)$ and for all g in $L^p(X, \mu)$. In section 3, we apply our results to Hilbert spaces. We first give two characterizations of Hilbert—Schmidt class operators, and then obtain a characterization of the semi-Carleman operators introduced by M. SCHREIBER [4]. Finally, we show that the Korotkov theorem for Carleman operators ([2], Theorem 1) remains valid even in nonseparable Hilbert space.

2. Throughout this section, all operators are bounded.

Theorem 2.1. *Let E be a Banach space such that either E has a separable strong dual E' or E is reflexive. For operators $T: L^p(X, \mu) \to E'$ with $1 < p < +\infty$ the following are equivalent:*

(i) *There exists a unique K in $L^{p'}(X, \mu; E')$ such that $Tg = \int_X g(x)K(x)d\mu$ for all g in $L^p(X, \mu)$.*

(ii) *There exists some non-negative γ in $L^{p'}(X, \mu)$ such that $\|Tg\| \leqq \int_X \gamma(x)|g(x)|d\mu$ for all g in $L^p(X, \mu)$.*

Note. The implication (i)$\Rightarrow$(ii) is trivial, as one may take $\gamma(x)=\|K(x)\|$. Moreover, the uniqueness of K in (i) is clear. For if there were some K and K' in $L^{p'}(X, \mu;\ E')$ such that $Tg=\int_X g(x)K(x)d\mu=\int_X g(x)K'(x)d\mu$ for all g in $L^p(X,\mu)$, then, in particular, $\int_A K(x)d\mu=\int_A K'(x)d\mu$ for all measurable set A with finite measure. Because the supports of K and K' are σ-finite measurable sets, we have therefore $K=K'$ in $L^p(X, \mu;\ E')$.

Theorem 2. 1 follows from Theorem 2 of [7] and the following lemma which may have some interest in its own right.

Lemma 2. 1. *Let E be a Banach space. Let $T^*: L^p(X, \mu)\to E'$ be the adjoint of $T: E\to L^{p'}(X, \mu)$, and let $\gamma\in L^{p'}(X, \mu)$, $\gamma\geqq 0$. Then the following are equivalent:*

(i) $|Tf(x)|\leqq\gamma(x)\|f\|$ *a.e. for f in E.*

(ii) $\|T^*g\|\leqq\int_X|g(x)|\gamma(x)d\mu$ *for all g in $L^p(X, \mu)$.*

Proof. *Case* 1: $\gamma(x)>0$ *a.e.* Form the finite measure space (X, ν) where $d\nu=\gamma^{p'}d\mu$. Let $M_\gamma: L^{p'}(X, \nu)\to L^{p'}(X, \mu)$, and $M_{\gamma^{p'/p}}: L^p(X, \nu)\to L^p(X, \mu)$ be the multiplication by γ and $\gamma^{p'/p}$, respectively. That is, $M_\gamma(g)=\gamma\cdot g$ and $M_{\gamma^{p'/p}}(h)=$ $=\gamma^{p'/p}\cdot h$ for g in $L^{p'}(X, \nu)$ and h in $L^p(X, \nu)$. Beause $\gamma(x)<0$ a.e., M_γ and $M_{\gamma^{p'/p}}$ are linear isomorphisms (onto), and $M_\gamma^{-1}=M_{\gamma^{-1}}$, $(M_{\gamma^{p'/p}})^{-1}=M_{\gamma^{-p'/p}}$. A simple computation shows that $(M_{\gamma^{p'/p}})^*=(M_\gamma)^{-1}$, hence $M_\gamma^*=M_{\gamma^{p'/p}}$. We now prove (ii)$\Rightarrow$(i). Write $T_p=T^*\circ M_{\gamma^{p'/p}}$. Then $T_p: L^p(X, \nu)\to E'$, and $\|T_pf\|\leqq\int_X|M_{\gamma^{p'/p}}(f)|\cdot\gamma d\mu$. Hence $\|T_pf\|\leqq\int_X|f(x)|d\nu=\|f\|_1$, where $\|\cdot\|_1$ denotes the L^1-norm of f. Since $L^p(X, \nu)$ is dense in $L^1(X, \nu)$, we can extend T_p to the whole of $L^1(X, \nu)$ without increasing its norm. Let $T_1: L^1(X, \nu)\to E'$ be the extension of T_p. Then $\|T_1\|\leqq 1$. We have the first one of the following commutative diagrams, from which the second one derives by taking adjoints:

$$\begin{tikzcd}
L^p(X,\nu) \arrow[r, "M_{\gamma^{p'/p}}"] \arrow[d, "i"'] \arrow[dr, "T_p"] & L^p(X,\mu) \arrow[d, "T^*"] \\
L^1(X,\nu) \arrow[r] & E'
\end{tikzcd}
\qquad
\begin{tikzcd}
L^{p'}(X,\nu) & L^{p'}(X,\mu) \arrow[l, "(M_{\gamma^{p'/p}})^*"'] \\
L^\infty(X,\nu) \arrow[u, "i^*"] & E'' \arrow[l, "T_1^*"] \arrow[u, "T^{**}"'] \arrow[ul, "T_p^*"']
\end{tikzcd}$$

Here i and i_λ are the natural embeddings, and $\|T_1^*\|=\|T_1\|\leqq 1$. If f in E, then $T^{**}f=Tf$. (Here we have identified E with a subset of E^{**} via the natural embedding.)

Therefore $i^*T_1^*(f)=(M_{\gamma^{p'/p}})^*T(f)=M_{\gamma^{-1}}(Tf)$. Hence $\|T_1^*f\|_\infty \leqq \|f\|$; it follows that $|T_1^*f(x)| \leqq \|f\|$ a.e. But $T_1^*f(x)=M_{\gamma^{-1}}Tf(x)=\gamma^{-1}(x)\cdot Tf(x)$. Therefore $|\gamma^{-1}(x)\cdot Tf(x)| \leqq \|f\|$ a.e. Hence $|Tf(x)|\leqq\gamma(x)\|f\|$ a.e. for f in E. This completes the proof of the implication (ii)⇒(i). The proof of (i)⇒(ii) is similar. We first consider the mapping $S_{p'}:E\to L^p(X,\nu)$ defined by $S_{p'}=M_{\gamma^{-1}}\circ T$. Then $|S_{p'}f(x)|\leqq\|f\|$ a.e. Let $i:L^\infty(X,\nu)\to L^{p'}(X,\nu)$ be the injection. Then $S_{p'}$ factors as $S_{p'}=i\circ S_\infty'$ where $S_\infty:E\to L^\infty(X,\nu)$ and $\|S_\infty f\|_\infty\leqq\|f\|$ where $\|\cdot\|_\infty$ denotes the L^∞-norm. Hence $\|S_\infty\|\leqq 1$. Therefore $S_\infty^*:M(X,\nu)\to E'$ is also a contraction where $M(X,\nu)$ is the dual of $L^\infty(X,\nu)$. It is clear that $i^*:L^p(X,\nu)\to M(X,\nu)$ is the natural injection which maps g into the finite measure (complex) $g d\nu$ for g in $L^p(X,\nu)$. Hence $\|S_\infty^*\circ i^*g\|\leqq\|i^*g\|$, and $\|i^*g\|=\int_X|g|d\nu$ for g in $L^p(X,\nu)$. Moreover, since $i\circ S_\infty=S_{p'}=M_{\gamma^{-1}}\circ T$, then $S_\infty^*\circ i^*=S_{p'}^*=T^*\circ(M_{\gamma^{-1}})^*=T^*\circ M_{\gamma^{p'/p}}$. It follows that $\|T^*\circ M_{\gamma^{p'/p}}g\|\leqq\int_X|g|d\nu$ for g in $L^p(X,\nu)$. If g is in $L^p(X,\mu)$, write $g=M_{\gamma^{p'/p}}(M_{\gamma^{p'/p}}g)$. Then $\|T^*g\|\leqq$ $\leqq\int_X|M_{\gamma^{-p'/p}}(g)|d\nu=\int_X\gamma(x)|g(x)|d\mu$. This proves (i)⇒(ii).

Case 2: *γ vanishes on a set of positive measure.* Let $Y=\{x;\ \gamma(x)>0\}$, and let (Y,μ) be the measure space obtained by restricting μ to Y. Let $j:L^{p'}(Y,\mu)\to L^{p'}(X,\mu)$ be the natural embedding. Then $j^*:L^p(X,\mu)\to L^p(Y,\mu)$ is the projection $g\to\chi_Y g$ where χ_Y is the characteristic function of Y. Then the operator T factors as $E\xrightarrow{T_Y}L^p(Y,\mu)\xrightarrow{j}L^p(X,\mu)$ if and only if T^* factors as $L^{p'}(X,\mu)\xrightarrow{j^*}L^{p'}(Y,\mu)\xrightarrow{T_Y^*}E'$. Now we apply the implication (i)⇒(ii) to the operators T_Y and T_Y^*, and complete the proof.

Proof of Theorem 2.1. We only need to prove that (ii) implies (i). Let $S:E\to L^{p'}(X,\mu)$ be the restriction of $T^*:E''\to L^{p'}(X,\mu)$ to E. Then $T=S^*$. By Lemma 2.1, we have $|Sf(x)|\leqq\gamma(x)\|f\|$ a.e. for f in E. By Theorem 2 of [7], we have $Tg=S^*g=\int_X K(x)g(x)d\mu$ for a unique K in $L^{p'}(X,\mu;\ E')$.

Remark. We note that, in Theorem 2.1, the existence of K does not depend upon the choice of those non-negative γ such that $\|Tg\|\leqq\int_X\gamma(x)|g(x)|d\mu$ for g in $L^p(X,\mu)$. The following lemma asserts that the function $\|K(\cdot)\|$ is the infimum of all those γ in the language of lattice theory. That is, $\|K(\cdot)\|=\Lambda\{\gamma\in L^{p'}(X,\mu);\ \|Tg\|\leqq\int_X\gamma(x)|g(x)|d\mu$ for all g in $L^p(X,\mu)\}$.

Lemma 2.2. *Let E be a Banach space. Let K be in $L^{p'}(X,\mu;\ E)$ and let γ be non-negative element in $L^{p'}(X,\mu)$ such that $\left\|\int_X K(x)g(x)d\mu\right\|\leqq\int_X\gamma(x)|g(x)|d\mu$ for all g in $L^p(X,\mu)$. Then $\|K(x)\|\leqq\gamma(x)$ a.e.*

Proof. Let $S:L^p(X,\mu)\to E$ be defined by $Sg=\int_X K(x)g(x)d\mu$. Then S is a bounded operator, and $S^*:E'\to L^{p'}(X,\mu)$ is given by $S^*f'(x)=\langle f',K(x)\rangle$ a.e. Furthermore, the proof for (ii)$\Rightarrow$(i) of Lemma 2. 1 proves that $|S^*f'(x)|\leqq\gamma(x)\|f'\|$ a.e. where the exceptional set of measure zero may depend upon f'. Hence $|\langle f',K(x)\rangle|\leqq\gamma(x)\|f'\|$ a.e. for f' in E'. Let N be the μ-null set such that $K(X\setminus N)$ is contained in a separable subset of E. Let $\{f_1,f_2,\dots,f_n,\dots\}$ be a countable dense subset of this subset of E. Let $\{f_1',f_2',\dots,f_n',\dots\}$ be the subset of E' such that $\|f_j'\|=1$ and $|\langle f_j',f_j\rangle|=\|f_j\|$ for each j. Then, if x is not in N, we have $\|K(x)\|=\sup_j|\langle f_j',K(x)\rangle|$. Let N_j be the μ-null such that $|\langle f_j',K(x)\rangle|\leqq\gamma(x)$ for all x not in N_j. Let $A=N\cup\left(\bigcup_{j=1}^{\infty}N_j\right)$. Then A is also a μ-null, and $\|K(x)\|\leqq\gamma(x)$ for all x not in A. This proves the lemma.

3. Let H be a Hilbert space. Let $S:H\to L^2(X,\mu)$ be a Hilbert—Schmidt class operator. For any orthonormal basis $\{f_\lambda\}$ of H, $\sum_\lambda\|Sf_\lambda\|^2$ is finite. There are at most countably many non-vanishing $\|Sf_\lambda\|^2$ in the above sum, say $Sf_{\lambda_j}\neq0$ $(j=1,2,3,\dots)$. Hence $\sum_{j=1}^{\infty}|Sf_{\lambda_j}(x)|^2<+\infty$ a.e. Let $K(x)=\sum_{j=1}^{\infty}Sf_{\lambda_j}(x)f_{\lambda_j}$. Then K is a strongly μ-measurable H-valued function such that $\int_X\|K(x)\|^2d\mu=\sum_{j=1}^{\infty}\|Sf_{\lambda_j}\|^2=\sum_\lambda\|Sf_\lambda\|^2=\|S\|_2^2$, where $\|S\|_2$ denotes the Hilbert—Schmidt norm of S. Furthermore, $Sf_{\lambda_j}(x)=\langle f_{\lambda_j},K(x)\rangle$ and hence $Sf(x)=\langle f,K(x)\rangle$ a.e. for f in H. Conversely, if K in $L^2(X,\mu;H)$ and $S:H\to L^2(X,\mu)$ is defined by $Sf(x)=\langle f,K(x)\rangle$ a. e. then it is clear that S is of Hilbert—Schmidt class with Hilbert—Schmidt norm $\|K\|$. This shows that every Hilbert—Schmidt class operator $S:H\to L^2(X,\mu)$ is of the form $Sf(x)=\langle f,K(x)\rangle$ a.e. for a unique K in $L^2(X,\mu;H)$. The above argument can also be found, for example, in [6], 2. 2 (1); we include it here for a later reference. The following characterization for Hilbert—Schmidt class operators first appeared in PERSSON's article ([3], Theorem 3) as a special case of his main result. It is also included in ([7], Corollary 3 and its following remark).

However, the following version is due to WEIDMANN ([6], 2. 10. Korollar) for separable Hilbert spaces.

Theorem 3. 1. *Let H be a Hilbert space. For a bounded operator T: $H\to L^2(X,\mu)$, the following are equivalent:*

(i) *T is of Hilbert—Schmidt class.*

(ii) *$|Tf(x)|\leqq\gamma(x)\|f\|$ a.e. for some non-negative γ in $L^2(X,\mu)$.*

(iii) *$Tf(x)=\langle f,K(x)\rangle$ a.e. for a unique K in $L^2(X,\mu;H)$.*

Moreover, $\|T\|_2=\|K\|$, where $\|T\|_2$ denotes the Hilbert—Schmidt norm of T, and $\|K\|$ denotes the norm of K in $L^2(X,\mu;H)$.

Proof. The argument given at the beginning of this section shows that (i) and (iii) are equivalent and $\|T\|_2=\|K\|$. By Theorem 1 of [7] we see that (ii) and (iii) are equivalent.

Dual to Theorem 3. 1, we have the following

Theorem 3. 2. *Let H be a Hilbert space. For a bounded operator $S:L^2(X,\mu)\to H$ the following are equivalent:*

(i) *S is of Hilbert—Schmidt class.*

(ii) *$\|Sg\|\leqq\int_X \gamma(x)\,|g(x)|d\mu$ for some non-negative γ in $L^2(X,\mu)$.*

(iii) *$Sg=\int_X K(x)g(x)d\mu$ for a unique K in $L^2(X,\mu;H)$.*

Moreover, $\|S\|_2=\|K\|$.

Proof. S is of Hilbert—Schmidt class if and only if $S^*:H\to L^2(X,\mu)$ is of Hilbert—Schmidt class. This is so if and only if $|S^*f(x)|\leqq\gamma(x)\|f\|$ a.e. for some $\gamma\geqq 0$ in $L^2(X,\mu)$. By Lemma 2. 1, the above inequality holds if and only if $\|Sg\|\leqq$ $\leqq\int_X\gamma(x)|g(x)|d\mu$. Hence (i) and (ii) are equivalent. (ii) and (iii) are equivalent by Theorem 2. 1. Furthermore, from Theorem 3. 1, we have $\|S^*\|_2=\|K\|$, but $\|S\|_2=$ $=\|S^*\|_2$. Hence $\|S\|_2=\|K\|$.

We now turn our attention to operators defined on a linear manifold of Hilbert space. Let H be a Hilbert space, and let K be a strongly μ-measurable H-valued function defined almost everywhere on X. Let $\mathfrak{D}=\{f\in H;\ \langle f, K(\cdot)\rangle\in L^2(X,\mu)\}$. Then $\mathfrak{D}$ is a linear manifold of H, but not necessarily dense in H. Let $\hat{\mathfrak{D}}=\{g\in L^2(X,\mu):\int_X\|K(x)\|\ |g(x)|d\mu<+\infty\}$. Then $\hat{\mathfrak{D}}$ is a dense linear manifold of $L^2(X,\mu)$ (cf. [4]).

Notice that $\hat{\mathfrak{D}}=\{g\in L^2(X,\mu);\ gK$ is Bochner integrable$\}$. Moreover, if $\hat{\mathfrak{D}}=L^2(X,\mu)$, then K is necessary in $L^2(X,\mu;H)$.

Following J. Weidmann [6] we call an operator $T:\mathfrak{D}_T\to L^2(X,\mu)$ a Carleman operator, if its domain $\mathfrak{D}_T$ is contained in $\mathfrak{D}$ and it can be written as $Tf(x)=\langle f, K(x)\rangle$ a.e. for f in $\mathfrak{D}_T$. An operator $S:\hat{\mathfrak{D}}_S\to H$ is called a semi-Carleman operator, if its domain $\hat{\mathfrak{D}}_S$ is contained in $\hat{\mathfrak{D}}$ and it can be written as $Sg=\int_X g(x)K(x)d\mu$ for g in $\hat{\mathfrak{D}}_S$.

We note that, when (X,μ) is σ-finite, and $H=L^2(X,\mu)$, then our definitions for Carleman and semi-Carleman operations coincide with the classical ones ([2] and [4]). For a detailed discussion of this see ([6], Section 5) or ([2], Lemma 1).

Theorem 3. 3. *Let (X,μ) be a σ-finite measure space. Let $S:\hat{\mathfrak{D}}_S\to H$ be an operator with dense domain $\hat{\mathfrak{D}}_S$ in $L^2(X,\mu)$. The following are equivalent:*

(i) *S is a semi-Carleman operator.*

(ii) *There exists a measurable function γ such that $0 \leqq \gamma(x) < +\infty$ a.e., $\hat{\mathfrak{D}}_S \subset \{g \in L^2 : \int_X \gamma(x)\, |g(x)| d\mu < +\infty\}$, and $\|Sg\| \leqq \int_X \gamma(x)\, |g(x)| d\mu$ for all g in $\hat{\mathfrak{D}}_S$.*

Proof. The implication (i)$\Rightarrow$(ii) is clear. We now prove (ii)$\Rightarrow$(i). Write $X = \bigcup_{n=1}^{\infty} A_n$, $A_n \subset A_{n+1}$ and $\mu(A_n) < +\infty$ for all n. Let $X_n = \{x \in A_n;\ \gamma(x) \leqq n\}$ for $n = 1, 2, \dots$. Then $X_n \subset X_{n+1}$, $\mu(X_n) < +\infty$ and $\mu(X \setminus \bigcup_n X_n) = 0$. Let μ_n be the restriction of μ to X_n, let $\hat{\mathfrak{D}}_n = \hat{\mathfrak{D}}_S \cap L^2(X_n, \mu_n)$, $\gamma_n = \chi_{X_n} \gamma$. Then γ_n is in $L^2(X_n, \mu_n)$, and $\hat{\mathfrak{D}}_n$ is dense in $L^2(X_n, \mu_n)$. Consider $S_n : \hat{\mathfrak{D}}_n \to H$; the restriction of S to $\hat{\mathfrak{D}}_n$. We have $\|S_n g\| \leqq \leqq \int_{X_n} \gamma_n(x)\, |g(x)| d\mu_n$ for g in $\hat{\mathfrak{D}}_n$. Then S_n admits a unique bounded extension to $L^2(X_n, \mu_n)$ which is also denoted by S_n. Moreover, the inequality $\|S_n g\| \leqq \leqq \int_X \gamma_n(x)\, |g(x)| d\mu_n$ holds for all g in $L^2(X_n, \mu_n)$. Therefore, by Theorem 3.2 $S_n g = = \int_X g(x) K_n(x) d\mu_n$ for a unique K_n' in $L^2(X_n, \mu_n; H)$. By Lemma 2.2 $\|K_n'(x)\| \leqq \gamma_n(x)$ a.e. Note that S_{n+1} extends S_n, using the uniqueness assertion once more we have $K_{n+1}'(x) = K_n'(x)$ a.e. on X_n. We now define K_n almost everywhere on X by putting $K_n(x) = K_n'(x)$ a.e. on X_n and $K_n(x) = 0$ for x not in X_n. Then K_n is μ-strongly measurable. Since $K_{n+1}(x) = K_n(x)$ a.e. on X_n, then $\lim_{n\to\infty} K_n(x)$ exists almost everywhere. Let $K(x) = \lim_{n\to\infty} K_n(x)$, then K is defined almost everywhere on X into H and K is also μ-strongly measurable. Moreover, $\|K(x)\| = \lim_{n\to\infty} \|K_n(x)\| \leqq \lim_{n\to\infty} \gamma_n(x) = \gamma(x)$ a.e. Hence $\int_X |g(x)|\, \|K(x)\| d\mu \leqq \int_X |g(x)| \gamma(x) d\mu < +\infty$ for all g in $\hat{\mathfrak{D}}_S$. Thus the integral $\int_X g(x) K(x) d\mu$ exists for g in $\hat{\mathfrak{D}}_S$. We have $\hat{\mathfrak{D}}_S \subset \{g \in L^2(X, \mu);\ \int |g(x)| \|K(x)\| d\mu < +\infty\}$. We want to show that $Sg = \int_X g(x) K(x) d\mu$ for g in $\hat{\mathfrak{D}}_S$. To see this, we let $g_n = \chi_{X_n} g$. Then $g_n(x) \to g(x)$ a.e. and $g_n \in L^2(X_n, \mu_n)$. $\|Sg - Sg_n\| \leqq \int_X \gamma(x)\, |g_n(x) - g(x)|\, d\mu \to 0$, by dominated convergence. But $Sg_n = S_n g_n = \int_{X_n} K_n(x) g_n(x) d\mu_n = \int_X K(x) g_n(x) d\mu$. On the other hand $\left\| \int_X K(x) g(x) d\mu - \int_X K(x) g_n(x) d\mu \right\| \leqq \int_X \|K(x)\|\, |g(x) - g_n(x)| d\mu \leqq \leqq \int_X \gamma(x)\, |g(x) - g_n(x)|\, d\mu \to 0$. Therefore $Sg = \int_X g(x) K(x) d\mu$ for g in $\hat{\mathfrak{D}}_S$. This completes the proof.

In 1965, V. B Korotkov gave a characterization for a Carleman operator on separable L^2-space which is what he called an integral operator of Carleman type (cf. [2], Theorem 1). His proof is based on the Dunford-Pettis Theorem. Recently, M. Schreiber and Gy. Targonski also obtained a new characterization for Carleman operators (cf. [5], Theorem 2. 1). However, J. P. Williams shows that the

Schreiber—Targonski theorem is a consequence of the Korotkov theorem (private communication). (See also [6], Satz 2. 11.) Using our result, we can prove that the Korotkov Theorem remains valid without the separability assumption on the Hilbert spaces.

Theorem 3. 4 (Korotkov [2], Theorem 1). *Let (X, μ) be a σ-finite measure space. Let $T:\mathfrak{D}_T \to L^2(X, \mu)$ be an operator with dense domain $\mathfrak{D}_T$ in a Hilbert space H. The following are equivalent conditions:*

(i) *T is a Carleman operator.*

(ii) *There exists a non-negative measurable function γ such that $\gamma(x) < +\infty$ a.e. and $|Tf(x)| \leqq \gamma(x)\|f\|$ a.e. for f in $\mathfrak{D}_T$.*

Proof. (i) clearly implies (ii). To prove (ii)$\Rightarrow$(i), we write $X = \bigcup_{n=1}^{\infty} A_n$, with $A_n \subset A_{n+1}$ and each A_n of finite measure. Let $X_n = \{x \in A_n;\ \gamma(x) \leqq n\}$. Then $\mu\left(X \setminus \bigcup_{n=1}^{\infty} X_n\right) = 0$, and $X_n \subset X_{n+1}$ and each X_n has finite measure. Let $E_n: L^2(X, \mu) \to L^2(X, \mu)$ be the projection on $L^2(X_n, \mu)$. Then $E_n \to 1$ strongly. Let $\gamma_n = \chi_{X_n}\gamma$, $\mu_n = \mu|_{X_n}$. Then γ_n in $L^2(X_n, \mu_n)$. Consider $E_n T:\mathfrak{D}_T \to L^2(X, \mu)$. Let $j: L^2(X_n, \mu_n) \to L^2(X, \mu)$ be the natural embedding. Then $E_n T$ factors as $\mathfrak{D}_T \xrightarrow{T_n} L^2(X_n, \mu_n) \xrightarrow{j} L^2(X, \mu)$ where $|T_n f(x)| \leqq \gamma_n(x)\|f\|$ a. e. (μ_n). T_n admits a unique bounded extension to H which is again written as T_n. By a standard density argument one can show that the extension T_n also has the property that $|T_n f(x)| \leqq \gamma_n(x)\|f\|$ a.e. (μ_n) for f in H. By Theorem 3. 1 T_n is of Hilbert—Schmidt class, and there is a unique K_n in $L^2(X_n, \mu_n; H)$ such that $T_n f(x) = \langle f, K_n'(x)\rangle$ a.e. By uniqueness again, we have $K_{n+1}'(x) = K_n'(x)$ a.e. on X_n. Let $K_n(x) = K_n'(x)$ a.e. on X_n and $K_n(x) = 0$ for x not in X_n. Then each K_n is strongly μ-measurable H-valued. Let $K(x) = \lim_{i\to\infty} K_n(x)$ a.e. Then K defines almost everywhere on X and is μ-measurable. Moreover $K(x) = K_n(x)$ a.e. on X_n. If f in $\mathfrak{D}_T$, then $Tf = \lim_{i\to\infty} E_n Tf$. But $E_n Tf = j \cdot T_n f$, so $(E_n T)f(x) = (T_n f)(x) = f, K_n(x)$ a. e. Therefore $Tf(x) = \lim_{i\to\infty} E_{n_i} Tf(x) = \lim_{i\to\infty}\langle f, K_{n_i}(x)\rangle = \langle f, K(x)\rangle$ a.e. This completes the proof.

4. Concluding remark. In the definition of a semi-Carleman operator, if we enlarge the linear manifold $\hat{\mathfrak{D}}$ to the linear manifold of $L^2(X, \mu)$ consisting of all g such that the H-valued function $x \to g(x)K(x)$ is weakly integrable in the sense of Pettis, where K is a μ-strongly measurable H-valued function. We may call an operator $T:\hat{\mathfrak{D}}_T \to H$ a weak semi-Carleman operator, if its domain $\hat{\mathfrak{D}}_T$ is contained in $\hat{\mathfrak{D}}$ and it can be written as $Tg = \int_X g(x)K(x)\,d\mu$ for g in $\hat{\mathfrak{D}}_T$, where the integral is the weak integral in the sense of Pettis. It is easy to see that, if $A: H \to L^2(X, \mu)$ is an everywhere defined Carleman operator (hence bounded), then

$A^*: L^2(X, \mu) \to H$ is a weak semi-Carleman operator. More than this, one can easily show that the adjoint of a densely defined Carleman operator is a closed extension of a weak semi-Carleman operator. It follows that the conditions $|Af(x)| \leqq \gamma(x)\|f\|$ and $\|A^*g\| \leqq \int_X \gamma(x)|g(x)|d\mu$, for some nonnegative measurable γ are not equivalent for the Carleman operator A. It would be interesting to give a characterization for a weak semi-Carleman operator.

Using theory of semi-ordered spaces, S. I. Ždanov (cf. [8], proof of Theorem 1) proved that the Korotkov inequality $|Tf(x)| \leqq \gamma(x)\|f\|$ a.e. is equivalent to that T maps every null sequence of vectors $\{f_n\}_{n=1}^{\infty}$ in H into a sequence $\{Tf_n\}_{n=1}^{\infty}$ in $L^2(X, \mu)$ such that $Tf_n(x) \to 0$ a.e. For a complete elementary proof of this see Weidmann ([6], Satz 2. 12). We do not know the answer to the following question:

What is the condition corresponding to the Ždanov theorem for a semi-Carleman operator and a Carleman operator respectively?

*

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Compléments à l'étude des opérateurs de classe C_0. II

Par BÉLA SZ.-NAGY à Szeged et CIPRIAN FOIAŞ à Bucarest

Un opérateur T dans un espace de Hilbert $\mathfrak{H}$ s'appelle de classe C_0 s'il est une contraction complètement non-unitaire et si $u(T)=0$ pour une fonction convenable $u \in H^\infty$, $u \not\equiv 0$. Dans [1] (théorème 2; *cf.* aussi [3], théorème 2) on a démontré que si tel T admet un vecteur cyclique, les opérateurs appartenant au commutant $(T)'$ sont de la forme $\varphi(T)$ (où $\varphi \in N_T$) et par conséquent $(T)'$ est commutatif. L'assertion inverse, notamment que pour $T \in C_0$ la commutativité de $(T)'$ entraîne l'existence d'un vecteur cyclique, n'y a été démontrée que dans la condition supplémentaire $\mu_T < \infty$. Le but de cette Note est d'écarter cette condition, de plus la démonstration qu'on va donner ne dépend pas du théorème 3 de [2] concernant le bicommutant $(T)''$.

Nous allons donc démontrer le suivant:

Théorème. *Soit T un opérateur dans $\mathfrak{H}$, de classe C_0 et avec $(T)'$ commutatif. T admet alors un vecteur cyclique.*

Démonstration. 1. Soit m la fonction minimum de T — et par suite $m^\sim$ celle de T^*. D'après la proposition 2 de [3] il existe des opérateurs T_1, T_2 de classe C_0, opérant dans des espaces $\mathfrak{H}_1$, $\mathfrak{H}_2$, selon les cas, tels que

$$T \succ S(m) \oplus T_1, \quad T^* \succ S(m^\sim) \oplus T_2^*;\text{[1]} \tag{1}$$

les fonctions minimum correspondantes m_{T_i} sont des diviseurs de m. Vu que $S(m)^*$ est unitairement équivalent à $S(m^\sim)$, [2] (1) entraîne

$$S(m) \oplus T_2 \succ T \succ S(m) \oplus T_1. \tag{2}$$

Donc il existe des quasi-affinités A_1, A_2 telles que

$$(S(m) \oplus T_2) A_2 = A_2 T, \quad T A_1 = A_1 (S(m) \oplus T_1). \tag{3}$$

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[1]) Pour les notations voir [2].

[2]) Voir [2], note 7.

Le produit $M=A_2A_1$ sera une quasi-affinité

$$M\colon\ \mathfrak{H}(m)\oplus\mathfrak{H}_1\to\mathfrak{H}(m)\oplus\mathfrak{H}_2$$

telle que

$$(S(m)\oplus T_2)M = M(S(m)\oplus T_1). \tag{4}$$

Désignons par $\mathfrak{X}$ l'ensemble des opérateurs

$$X\colon\ \mathfrak{H}(m)\oplus\mathfrak{H}_2\to\mathfrak{H}(m)\oplus\mathfrak{H}_1$$

vérifiant la relation

$$(S(m)\oplus T_1)X = X(S(m)\oplus T_2). \tag{5}$$

Comme (3) et (5) entraînent $A_1XA_2\in(T)'$ pour $X\in\mathfrak{X}$, et puisque $(T)'$ est commutatif, il s'ensuit que

$$A_1XA_2\cdot A_1X'A_2 = A_1X'A_2\cdot A_1XA_2$$

et par conséquent

$$XMX' = X'MX \quad \text{pour} \quad X, X'\in\mathfrak{X}. \tag{6}$$

2. Un exemple banal pour $X\in\mathfrak{X}$ est l'opérateur X_0 défini par

$$X_0(h\oplus h_2) = h\oplus 0 \qquad (h\oplus h_2\in\mathfrak{H}(m)\oplus\mathfrak{H}_2). \tag{7}$$

D'autres exemples s'obtiennent de la manière suivante.

Fixons $h_{10}\in\mathfrak{H}_1$ et considérons la restriction T_{10} de T_1 au sous-espace invariant

$$\mathfrak{H}_{10} = \bigvee_{n\geq 0} T_1^n h_{10}.$$

T_{10} est de classe C_0 et cyclique, donc il existe un diviseur intérieur m_1 (de m_{T_1} et par conséquent) de m tel que $S(m_1)\prec T_{10}$ (*cf.* [3], théorème 2). Il existe donc une quasi-affinité

$$B\colon\ \mathfrak{H}(m_1)\to\mathfrak{H}_{10}$$

vérifiant

$$T_{10}B = BS(m_1). \tag{8}$$

D'autre part il existe un opérateur

$$P\colon\ \mathfrak{H}(m)\to\mathfrak{H}(m_1)$$

tel que

$$\overline{P\mathfrak{H}(m)} = \mathfrak{H}(m_1) \quad \text{et} \quad S(m_1)P = PS(m).\,^{3)} \tag{9}$$

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[3]) En posant $m=m_1m_1'$, $P'=m_1'(S(m))$ et $\mathfrak{H}'=\overline{P'\mathfrak{H}(m)}$, la restricition T' de $S(m)$ au sous-espace invariant $\mathfrak{H}'$ est de classe $C_0(1)$ et a la fonction minimum m_1; *cf.* [3], note [9]). Donc il existe un opérateur unitaire $W\colon \mathfrak{H}'\to\mathfrak{H}(m_1)$ tel que $S(m_1)W=WT'$, et il n'y aura qu'à poser $P=WP'$.

On déduit de (8) et (9) que l'opérateur

$$C = BP\colon\ \mathfrak{H}(m) \to \mathfrak{H}_1$$

vérifie les relations

(10) $$\overline{C\mathfrak{H}(m)} = \mathfrak{H}_{10} \ni h_{10} \quad \text{et} \quad T_1 C = CS(m).$$

En utilisant (8) et (9) on déduit aussi que l'opérateur X_{10} défini par

(11) $$X_{10}(h \oplus h_2) = 0 \oplus Ch \qquad (h \oplus h_2 \in \mathfrak{H}(m) \oplus \mathfrak{H}_2)$$

appartient à $\mathfrak{X}$.

Un type dual d'opérateurs dans $\mathfrak{X}$ peut être construit de la manière suivante. En passant aux adjoints il dérive de (2) que

$$S(m)^* \oplus T_1^* \succ T^* \succ S(m)^* \oplus T_2^*.$$

En fixant un vecteur $h_{20} \in \mathfrak{H}_2$ on montre tout comme ci-dessus (et en rappelant que $S(m)^*$ est unitairement équivalent à $S(m^\sim)$) qu'il existe un opérateur

$$D\colon\ \mathfrak{H}(m) \to \mathfrak{H}_2$$

tel que

(11) $$\overline{D\mathfrak{H}(m)} \ni h_{20} \quad \text{et} \quad T_2^* D = D\, S(m)^*,$$

et par conséquent

(13) $$S(m) D^* = D^* T_2.$$

Pour l'opérateur X_{20} défini par

(14) $$X_{20}(h \oplus h_2) = D^* h_2 \oplus 0 \qquad (h \oplus h_2 \in \mathfrak{H}(m) \oplus \mathfrak{H}_2)$$

on déduit de (13) que $X_{20} \in \mathfrak{X}$.

3. Cela étant, appliquons (6) d'abord aux opérateurs X_0 et X_{10}, le dernier dérivant d'un vecteur fixé $h_{10} \in \mathfrak{H}_1$. Puisque les ensembles de valeurs de X_0 et X_{10} sont évidemment orthogonaux l'un à l'autre, l'égalité $X_0 M X_{10} = X_{10} M X_0$ entraîne $X_0 M X_{10} = 0$, donc

$$X_0 M(0 \oplus Ch) = X_0 M X_{10}(h \oplus h_2) = 0$$

pour tout $h \oplus h_2 \in \mathfrak{H}(m) \oplus \mathfrak{H}_2$. Vu (10) cela entraîne aussi

$$X_0 M(0 \oplus h_{10}) = 0;$$

en vertu de (7) cela veut dire que

(15) $$M(0 \oplus h_{10}) = 0 \oplus h_{20}$$

pour un certain vecteur $h_{20} \in \mathfrak{H}_2$.

Envisageons l'opérateur X_{20} construit en partant de ce vecteur h_{20} et l'opérateur X_{10} ci-dessus, engendré par h_{10}. Par les définitions (11) et (14) les ensembles de valeurs de X_{10} et X_{20} sont orthogonaux l'un à l'autre, donc l'égalité $X_{10}MX_{20}= =X_{20}MX_{10}$ entraîne $X_{20}MX_{10}=0$. On a donc

$$X_{20}M(0\oplus Ch) = X_{20}MX_{10}(h\oplus h_2) = 0$$

pour tout $h\oplus h_2\in\mathfrak{H}(m)\oplus\mathfrak{H}_2$. Vu (10) cela entraîne aussi

$$X_{20}M(0\oplus h_{10}) = 0. \tag{16}$$

En vertu des relations (14), (15), (16) et (12) on a

$$D^*h_{20}\oplus 0 = X_{20}(0\oplus h_{20}) = X_{20}M(0\oplus h_{10}) = 0,$$

d'où

$$D^*h_{20} = 0, \quad h_{20}\perp\overline{D\mathfrak{H}(m)}\ni h_{20}, \quad h_{20} = 0.$$

Toujours par (15) on a donc $M(0\oplus h_{10})=0$, et comme M est injectif, il résulte que $h_{10}=0$.

Comme pour h_{10} on a pu choisir un vecteur quelconque de $\mathfrak{H}_1$, on conclut que $\mathfrak{H}_1=\{0\}$, et par conséquent $T\succ S(m)$; puisque $S(m)$ admet un vecteur cyclique, il en est de même de T. Cela achève la démonstration.

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Accretive operators: Corrections

By BÉLA SZ.-NAGY and CIPRIAN FOIAŞ

1. In Chapter IV of the monograph [1] we made a statement (Lemma 5. 2) which in general is false; the error stemmed from an incorrect use of Schwarz's inequality for non necessarily symmetric real bilinear forms[1]). However, a somewhat weaker statement (Lemma 5. 2 below) is sufficient for the concluding part of the proof of Proposition 5. 5 (i.e., Langer's uniqueness theorem for the accretive nth roth of a maximal accretive operator). Lemma 5. 3 (which is also needed in the proof of Proposition 5. 5) can be given an independent proof.

The two lemmas and their proofs should read as follows.

Lemma 5. 2. *Let A be a linear operator in the Hilbert space $\mathfrak{H}$, densely defined and such that*

$$|\arg (Ah, h)| \leqq \alpha\pi/2 \quad \textit{for some} \quad \alpha (0 \leqq \alpha \leqq 1) \quad \textit{and all} \quad h \in \mathfrak{D}(A).$$

If $\alpha < 1$ then $(Ah, h) = 0$ implies $h = 0$.

Proof. The binary forms $(g|h)_{\pm} = \mathrm{Re}[e^{\pm i(1-\alpha)\pi/2}(Ag, h)]$ on $\mathfrak{D}(A)$ are bilinear with respect to real coefficients and satisfy $(h|h)_{\pm} \geqq 0$. Therefore the Schwarz type inequality

$$\left|\tfrac{1}{2}(g|h)_{\pm} + \tfrac{1}{2}(h|g)_{\pm}\right|^2 \leqq (g|g)_{\pm} \cdot (h|h)_{\pm} \tag{5. 12}$$

holds and as a consequence $(Ah, h) = 0$ implies

$$\mathrm{Re}\,\{e^{\pm i(1-\alpha)\pi/2}[(Ag, h) + (Ah, g)]\} = 0 \quad \text{for all} \quad g \in \mathfrak{D}(A).$$

Suppose $\alpha < 1$. Then $\pm(1-\alpha)\pi/2$ are not congruent modulo π, and hence we infer that

$$(Ag, h) + (Ah, g) = 0 \quad \text{for all} \quad g \in \mathfrak{D}(A).$$

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[1]) We are indebted for this remark to Professors RICK CAREY, J. E. KERLIN and A. L. LAMBERT at the University of Kentucky in Lexington, U.S.A., and UWE BÖCKER at the University of Frankfurt/Main, Germany.

This holds for ig as well as for g so we also have

$$(Ag, h)-(Ah, g) = 0,$$

and therefore $(Ah, g)=0$ for all $g \in \mathfrak{D}(A)$. As $\mathfrak{D}(A)$ is dense in $\mathfrak{H}$, we conclude that $Ah=0$.

Lemma 5. 3. *For any closed accretive operator A in $\mathfrak{H}$, the set*

$$\mathfrak{N} = \{g\colon\ g \in \mathfrak{D}(A),\ Ag = 0\}$$

is a subspace of $\mathfrak{H}$ reducing A.

Proof. As A is linear and closed, the set $\mathfrak{N}$ is also linear and closed, i.e. a subspace of $\mathfrak{H}$. For $h \in \mathfrak{D}(A)$ and $g \in \mathfrak{N}$ we have $(Ah, g)=0$ as a consequence of inequality (5. 12) for $\alpha=1$. Thus if $h \in \mathfrak{D}(A)$ then $P_{\mathfrak{N}}Ah=0$, where $P_{\mathfrak{N}}$ denotes orthogonal projection onto $\mathfrak{N}$. On the other hand, $AP_{\mathfrak{N}}h=0$ obviously holds for every $h \in \mathfrak{H}$. Thus we have $P_{\mathfrak{N}}A \subset AP_{\mathfrak{N}}$, and hence $\mathfrak{N}$ reduces A.

2. We use this opportunity to correct the Notes to Chapter IV of [1]. There it is asserted that Proposition 4. 2 (on the simultaneous extension of some dually coupled accretive operators) is new. Although it was independently found by the authors, the result is essentially contained in Ref. [2].

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Some theorems on unitary ϱ-dilations of Sz.-Nagy and Foiaş

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Introduction. Sz.-Nagy and Foiaş introduced, for each fixed $\varrho>0$, the class C_ϱ of operators T on a given complex Hilbert space H for which there exist a Hilbert space K containing H as a subspace and a unitary operator U on K satisfying the following relation:

(1) $$T^n = \varrho \cdot PU^n \qquad (n = 1, 2, \ldots)$$

where P is the orthogonal projection of K on H; this unitary operator U is called a unitary ϱ-dilation of T.

It is well known that $C_1 = \{T: \|T\| \leqq 1\}$ ([7]) and that $C_2 = \{T: w(T) \leqq 1\}$ ([1]), where $w(T)$ denotes the numerical radius of T i.e.

(2) $$w(T) = \sup |(Th, h)| \quad \text{for} \quad h \in H, \|h\| = 1.$$

Sz.-Nagy and Foiaş have characterized C_ϱ for general $\varrho>0$. One of their results is:

Theorem A ([8]). *An operator T on H belongs to the class C_ϱ ($\varrho \geqq 2$) if and only if it satisfies the following conditions:*

$(*)$ $$\|(\mu I - T)^{-1}\| \leqq \frac{1}{|\mu|-1} \begin{cases} \text{for} \quad 1<|\mu|<\infty & \text{if } \varrho = 2, \\ \text{for} \quad 1<|\mu| \leqq \dfrac{\varrho-1}{\varrho-2} & \text{if } \varrho>2, \end{cases}$$

$(**)$ *T has its spectrum in the closed unit disc.*

In [6] J. A. R. Holbrook introduced the functions $w_\varrho(T)$ defined on the space $B(H)$ of all operators on H as follows

(3) $$w_\varrho(T) = \inf\left\{u: u>0, \ \frac{1}{u} T \in C_\varrho\right\};$$

in particular, we have $w_2(T) = w(T)$, $w_1(T) = \|T\|$, and

(4) $$C_\varrho = \{T: w_\varrho(T) \leqq 1\}.$$

The following theorem holds:

Theorem B ([6]). *$w_\varrho(T)$ has the following properties:*

(i) $w_\varrho(T)<\infty$;

(ii) $w_\varrho(T)>0$ *unless* $T=0$, *in fact* $w_\varrho(T)\geqq\frac{1}{\varrho}\|T\|$;

(iii) $w_\varrho(zT)=|z|w_\varrho(T)$;

(iv) $w_\varrho(T)$ *is a norm whenever* $0<\varrho\leqq 2$;

(v) $w_\varrho(T)$ *is continuous and non-increasing as a function of* ϱ; *moreover,* $r(T)\leqq w_\varrho(T)$ *for* $\varrho>0$ *and* $\lim_{\varrho\to\infty} w_\varrho(T)=r(T)$, *where* $r(T)$ *is the spectral radius of* T;

(vi) *the "power inequality" holds:* $w_\varrho(T^k)\leqq (w_\varrho(T))^k$ $\quad (k=1,2,\ldots)$.

In [2] and [8] there are given examples of power bounded operators which are not contained in any of the classes C_ϱ.

1. The theorems and their corollaries

Theorem 1. *If* $T^2=T$ *and* $T\in C_\varrho$, *then* T *is a projection.*

Theorem 2. *If* $T^k=T$ *for some positive integer* $k\geqq 2$ *and* $T\in C_\varrho$, *then* T *is the direct sum of a zero operator and of a unitary operator, i.e.* T *is normal and partially isometric.*

Corollary 1 ([4]). *If* T *is an idempotent operator that satisfies any of the following conditions*

(i) T *is a contraction;*

(ii) T *is a numerical radius contraction* $(w(T)\leqq 1)$,

(iii) T *has equal norm and spectral radius (normaloid* [5]),

(iv) T *has equal numerical and spectral radius (spectraloid* [5]),

then T *is an orthogonal projection.*

Corollary 2 ([4]). *If* $T^k=T$ *for some positive integer* $k\geqq 2$ *and satisfies any of the conditions* (i)—(iv) *in Corollary* 1, *then* T *is the direct sum of a zero operator and of a unitary operator, i.e.* T *is normal and partially isometric.*

Corollary 3. *If* $T^k=T$ *for some positive integer* $k\geqq 2$ *and* $\|T\|>1$, *then* T *is not contained in any of the classes* C_ϱ.

Corollary 3 gives another simple examples of power bounded operators which are not contained in any of the classes C_ϱ.

Proof of Theorem 1. By the idempotency of T, $R(T)$ (the range of T) coincides with null space of $I-T$, so that $R(T)$ is a closed subspace of H. Let P_1 and P_2 denote the orthogonal projections of H onto $R(T)$ and $R(T)^\perp$, respectively.

We consider the matrix of T with respect to the decomposition $H=R(T)\oplus R(T)^{\perp}$ i.e.

$$T=\begin{pmatrix} P_1TP_1 & P_1TP_2 \\ P_2TP_1 & P_2TP_2\end{pmatrix}=\begin{pmatrix} I & S \\ O & O\end{pmatrix},\quad (\mu I-T)^{-1}=\begin{pmatrix} \dfrac{1}{\mu-1}I & \dfrac{1}{\mu(\mu-1)}S \\ O & \dfrac{1}{\mu}I\end{pmatrix}.$$

We suppose that T is not a projection, that is, $S\neq 0$. Then

$$\|(\mu I-T)^{-1}\|=\sqrt{\frac{1}{|\mu-1|^2}+\frac{\|S\|^2}{|\mu(\mu-1)|^2}}>\frac{1}{|\mu-1|};$$

by taking μ real with $1<\mu\leqq\dfrac{\varrho-1}{\varrho-2}$, we obtain

$$\|(\mu I-T)^{-1}\|>\frac{1}{|\mu-1|}=\frac{1}{|\mu|-1}.$$

Hence T does not satisfy condition (*) for any $\varrho\geqq 2$. Since C_ϱ is a non-decreasing function of ϱ, we have $T\notin C_\varrho$ for any $\varrho>0$. This contradiction proves Theorem 1.

Theorem 3. *If $T^k=T$ for some postive integer $k\geqq 2$ and $T\in C_\varrho$, then T^{k-1} is a projection.*

Proof. We have $T^{2(k-1)}=T^{k-2}T^k=T^{k-2}T^1=T^{k-1}$, which implies that T^{k-1} is an idempotent operator. Hence by (4) and the power inequality for $w_\varrho(T)$ we have $w_\varrho(T^{k-1})\leqq(w_\varrho(T))^{k-1}\leqq 1$ so that $T^{k-1}\in C_\varrho$; thus T^{k-1} is a projection by Theorem 1.

Proof of Theorem 2. It is sufficient to consider the case that $T^k=T$ and $T\in C_\varrho$, where $k\geqq 2$ and $\varrho\geqq 1$. By Theorem 3, $P=T^{k-1}$ is a projection. Set $M=R(P)$. The relation $T=TP=PT$ implies that M reduces T and that T is zero on $M^{\perp}$.

On the other hand, $T_1=T|M$ satisfies $T_1^{k-1}=P|M=I_M$ and $w_\varrho(T_1)\leqq 1$. Thus we have $T_1^{-1}=T_1^{k-2}$. By the power inequality for $w_\varrho(T)$

$$w_\varrho(T_1^{-1})=w_\varrho(T_1^{k-2})\leqq(w_\varrho(T_1))^{k-2}\leqq 1,$$

whence we have $w_\varrho(T_1)\leqq 1$ and $w_\varrho(T_1^{-1})\leqq 1$ for $\varrho\geqq 1$, therefore T_1 is unitary ([9]). Consequently T is the direct sum of zero operator and of a unitary operator, that is to say, T is normal and partially isometric.

Corollaries 1 and 2 follow from Theorems 1 and 2 and from the fact that $w_\varrho(T)$ is a continuous and non-increasing function of ϱ. Corollary 3 is obvious by Theorem 2.

If $T^2=I$ and $T\in C_\varrho$, then T is unitary ([9]). Hence we remark that if $T^2=I$ and $\|T\|>1$, then $T\notin C_\varrho$ for any ϱ, in fact there are given two concrete examples in [2] and [8], which satisfy $T^2=I$ and $T\notin C_\varrho$ for any ϱ.

2. "ϱ-oid" operators

Definition 1 ([3]). An operator T will be called "*ϱ-oid*" if

$$w_\varrho(T^k)=(w_\varrho(T))^k \qquad (k=1, 2, \ldots);$$

1-oid and 2-oid operators are normaloid and spactraloid, respectively ([5]).

Theorem C ([3]), *For each* $\varrho \geqq 1$,

$$w_\varrho(T)=r(T) \quad \text{if and only if} \quad w_\varrho(T^k)=(w_\varrho(T))^k \qquad (k=1, 2, \ldots).$$

For each $0<\varrho<1$ *there exists no non-zero "ϱ-oid" operator which is included in the class of normaloids* ([3]).

By the power inequality $w_\varrho(T^k)\leqq(w_\varrho(T))^k$ $(k=1, 2, \ldots)$, Theorems 1 and 2, we have the following corollaries.

Corollary 4. *If T is "ϱ-oid" and $T^2=T$, then T is a projection.*

Corollary 5. *If T is "ϱ-oid "and $T^k=T$ for some positive integer $k\geqq 2$, then T is the direct sum of zero and a unitary operator, that is to say, T is normal and partially isometric.*

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The spectral theorem for real Hilbert space

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Introduction

The main purpose of this paper is to prove a spectral theorem for bounded normal operators in real Hilbert space. The self-adjoint case will follow as a corollary and is almost exactly the same as in the complex case. (However, the self-adjoint case for real Hilbert space is implicit in [3, pp. 269—276]). For normal operators, the theorem differs significantly from that for the complex case.

We begin by giving an example of a bounded normal operator in a real Hilbert space, which will turn out to be "essentially" the only example of a bounded normal operator in real Hilbert space. Consider $L_2(\mu)$ where μ is a measure with compact support defined on the Borel sets of the Euclidean plane. Further suppose that μ is symmetric about the x-axis, i.e., $\mu(e)=\mu(e^*)$ for each Borel set e, where e^* is the reflection of e about the x-axis. Then $L_2(\mu) = H_e \oplus H_0$, where H_e consists of the $L_2(\mu)$ functions that are symmetric (even) about the x-axis, and H_0 consists of the $L_2(\mu)$ functions that are anti-symmetric (odd) about the x-axis. Consider f in $L_2(\mu)$ as a function of (r, θ), and define the operator $T=T(\mu)$ on $L_2(\mu) = H_e \oplus H_0$ by

$$Tf = \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} f_e \\ f_0 \end{pmatrix}.$$

1. The Spectral Theorem

Let H be a real Hilbert space and let A be an everywhere defined and bounded operator from H into H, and in particular let A be normal ($AA^*=A^*A$). Let $\bar{H}$ be the complexification of H, with elements $[x, y]$ ($x, y \in H$), and inner product $\langle [x, y], [t, z] \rangle = (x, t) - i(x, z) + i(y, t) + (y, z)$. Define $\bar{A}v = [Ax, Ay]$ if $v = [x, y]$. Then $\bar{A}$ is linear, bounded, and normal, with $\bar{A}^* v = [A^* x, A^* y]$ and $\|\bar{A}\| = \|A\|$.

*) The author would like to thank Professors KARL GUSTAFSON and ARLAN RAMSAY for valuable suggestions and stimulating discussions concerning this paper. The author would especially like to thank Prof. GUSTAFSON for suggesting this problem to the author.

By the spectral theorem in complex Hilbert space, $\bar{A} = \int_{\sigma(A)} \lambda d\bar{E}$, where $\bar{E}$ is a self-adjoint measure, and $\sigma(\bar{A})$ is the spectrum of $\bar{A}$ in $\bar{H}$. Let D be the disk in the plane with center 0 and radius $\|A\|$; then D contains $\sigma(\bar{A})$.

We now compute $\bar{E}(e)$ for each Borel set e contained in D. If e is a compact set then χ_e (characteristic function of e) is the point limit of a bounded sequence of polynomials $p_n(z)$ in z and $\bar{z}$, and hence $\bar{E}(e)v = \lim p_n(\bar{A})v$ for each vector v in $\bar{H}$. But $(a+bi)(\bar{A})^n(\bar{A}^*)^m v = [aA^n A^{*m}x - bA^n A^{*m}y, aA^n A^{*m}y + bA^n A^{*m}x]$. Hence we deduce that $\bar{E}(e)$ is of the form $\bar{E}(e)v = [E_1(e)x - E_2(e)y, E_2(e)x + E_1(e)y]$ where $E_1(e)$, $E_2(e)$ are bounded operators.

Let S be the collection of Borel subsets e of D such that $\bar{E}(e)(v) = [\Phi_1 x - \Phi_2 y, \Phi_2 x + \Phi_1 y]$ for all $v = [x, y] \in \bar{H}$, where Φ_1 and Φ_2 are bounded operators. From the above S contains the compact sets, and it is easily verified that S is a σ-ring. (To show S is closed under complements one uses $\bar{E}(D) = I$, and to show S is closed under intersections one uses $\bar{E}(e_1 \cap e_2) = \bar{E}(e_1)\cdot\bar{E}(e_2)$. To prove S is closed under monotone limits use the fact that $\bar{E}(e)v = \lim \bar{E}(e_n)v$, where one considers v of the form $[x, 0]$ and of the form $[0, y]$, and the uniform boundedness principle.)

So for each Borel set e there exist unique bounded operators $E_1(e)$, $E_2(e)$ such that $\bar{E}(e)v = [E_1(e)y, E_2(e)x - E_2(e)x + E_1(e)y]$. We have $(\bar{E}(e))^* = \bar{E}(e)$, which is equivalent to saying that $(E_1(e))^* = E_1(e)$, and $(E_2(e))^* = -E_2(e)$, i.e. that $E_1(e)$ is self-adjoint and $E_2(e)$ is skew-symmetric. Since $E_2(e)$ is skew-symmetric, $(E_2(e)x, x) = 0$ for each x. Also, $\langle \bar{E}(e)v, v\rangle \geqq 0$ which implies $\mu(e) = (E_1(e)x, x) \geqq 0$ for all x. $(\bar{E}(e)v, w)$ is a regular Borel measure for each v, w which implies that μ above is a regular non-negative Borel measure. Also, since $\left\|\left(\sum_{i=1}^{n} \lambda_i \bar{E}(e_i)\right)v\right\| \leqq \max|\lambda_i|\cdot\|v\|$ for any finite partition $\{e_1, e_2, \ldots, e_n\}$ of D, one has $\|\Sigma\lambda_i E_j(e_i)x\| \leqq \max|\lambda_i|\cdot\|x\|$ for $j = 1, 2$, so each E_j is of bounded variation. This implies $\int f dE_j$ exists as a bounded operator from H into H for every bounded Borel measurable function f.

From the identity $\bar{E}(e_1 \cap e_2) = \bar{E}(e_1)\cdot\bar{E}(e_2)$ one obtains the identities

1) $E_1(e_1 \cap e_2) = E_1(e_1)E_1(e_2) - E_2(e_1)E_2(e_2)$,

2) $E_2(e_1 \cap e_2) = E_2(e_1)E_1(e_2) + E_1(e_1)E_2(e_2)$.

Since $\bar{E}(D) = I$, we have $E_1(D) = I$ and $E_2(D) = 0$. Also, $E_j\left(\bigcup_{i=1}^{\infty} e_i\right)x = \sum_i E_j(e_i)x$ for each x in H and for $j = 1, 2$ if the $\{e_i\}$ are disjoint. So E_1, E_2 are of bounded variation and countable additive in the strong operator topology.

From the spectral theorem we have $\bar{A}^n\bar{A}^{*m} = \int \lambda^{n-m} d\bar{E}$. Writing

$$\lambda = r(\cos\theta + i\sin\theta),$$

and expanding $\sum_{i=1}^{n} \lambda_i^n \bar{\lambda}_i^m \bar{E}(e_i)$ component-wise and taking a limit, we get $A^n A^{*m} = = \int r^{n+m} \cos(n-m)\theta dE_1 - \int r^{n+m} \sin(n-m)\theta dE_2$, and $\int r^{n+m} \sin(n-m)\theta dE_1 = = -\int r^{n+m} \cos(n-m)\theta dE_2$.

Lemma 1. *$E_1(e^*) = E_1(e)$ and $E_2(e^*) = -E_2(e)$.*

Proof. $0 = \int r^{n+m} \cos(n-m)\theta d(E_2 x, x) = -\int r^{n+m} \sin(n-m)\theta d\mu$, where $\mu(e) = (E_1(e)x, x)$. Let f be continuous on D and $f(r, -\theta) = -f(r, \theta)$. Given $\varepsilon > 0$, by the Stone—Weierstrass theorem there exists a trigonometric polynomial $p(r, \theta)$ such that $|f(r, \theta) - p(r, \theta)| < \varepsilon/2$ in D, where $p(r, \theta) = \Sigma a_{n,m} r^{n+m} \sin(n-m)\theta + + \Sigma b_{n,m} r^{n+m} \cos(n-m)\theta$. Substituting $-\theta$ into the above inequality and adding show $|\Sigma b_{n,m} r^{n+m} \cos(n-m)\theta| < \varepsilon/2$. This implies $|f(r, \theta) - \Sigma a_{n,m} r^{n+m} \sin(n-m)\theta| < \varepsilon$. This gives $\int f(r, \theta) d\mu = 0$. If e is a compact set lying entirely in the upper halfplane then there exists a bounded sequence $\{f_n\}$ of continuous functions converging pointwise to χ_e and vanishing off the upper half-plane. Define g_n to equal f_n in the upper half-plane and $-f_n(r, \theta)$ in the lower half-plane. Then $\int g_n d\mu = 0$, but g_n converges pointwise to $\chi_e - \chi_{e^*}$, so by the dominated convergence theorem $\int (\chi_e - \chi_{e^*}) d\mu = 0$, or $\mu(e) = \mu(e^*)$. Let S be the collection of Borel sets e in D, lying in the upper half-plane and such that $\mu(e) = \mu(e^*)$. One can show that S is a σ-ring containing the compact sets, so that $\mu(e) = \mu(e^*)$ for each Borel set e lying in the upper half-plane. From this it easily follows that $\mu(e) = \mu(e^*)$ for every Borel set. e Thus $(E_1(e)x, x) = = (E_1(e^*)x, x)$ for each x, which implies $E_1(e) = E_1(e^*)$ since $E_1(e)$ is self-adjoint.

From the identity $\int r^{n+m} \cos(n-m)\theta dE_2 = -\int r^{n+m} \sin(n-m)\theta dE_1$, and since E_1 is symmetric about the x-axis and $r^{n+m} \sin(n-m)\theta$ is antisymmetric, we have $\int r^{n+m} \cos(n-m)\theta dE_2 = 0$; hence $\int r^{n+m} \cos(n-m)\theta d\nu = 0$, where $\nu(e) = (E_2(e)x, y)$. An argument similar to the above shows $\nu(e) = -\nu(e^*)$, i.e., $E_2(e) = -E_2(e^*)$.

Definition. Let (E_1, E_2) be called a spectral pair provided 1) E_1 and E_2 are of finite variation and countably additive in the strong operator topology; 2) $E_1(e)$ is self-adjoint and $E_2(e)$ is anti-symmetric for each Borel set e; 3) $E_1(e) = = E_1(e^*)$, and $E_2(e) = -E_2(e^*)$ for each Borel set e; 4) $E_1(e_1 \cap e_2) = E_1(e_1)E_1(e_1) - - E_2(e_1)E_2(e_2)$, and $E_2(e_1 \cap e_2) = E_2(e_1)E_1(e_2) + E_1(e_1)E_2(e_2)$; 5) $E_1(D) = I$, $E_2(D) = 0$.

Summarizing the above:

Theorem 1. *If A is a bounded normal operator on a real Hilbert space then there exists a unique spectral pair (E_1, E_2) such that $Ax = \int r \cos\theta d E_1 x - \int r \sin\theta d E_2 x$ for all $x \in H$.*

Proof. The only remaining item to check is the uniqueness of the pair. Suppose (E_1', E_2') is another such spectral pair. Let $\bar{E}'(e)v=[E_1'(e)x-E_2'(e)y, E_2'(e)x+E_1'(e)y]$. One can show directly that $\bar{E}'$ is a spectral measure and that $\bar{A}=\int \lambda d\bar{E}'$. By the spectral theorem in the complex case $\bar{E}'=\bar{E}$, and this implies $E_1'=E_1$, $E_2'=E_2$.

2. Spectral Representation

Theorem 1 will now be used to prove a spectral representation theorem, i.e., we will show that H is the orthogonal direct sum of closed subspaces $\{H_\alpha\}$ where each H_α is isometrically isomorphic to an $L_2(\mu_\alpha)$ space and $A|H_\alpha$ is characterized on $L_2(\mu_\alpha)$ as the operator T_α described in the introduction.

Suppose there exists a cyclic vector $x\in H$ such that the linear span of the vectors of the form $A^n A^{*m}x$ is dense in H. Let $\mu(e)=(E_1(e)x, x)$. Then μ is a non-negative regular Borel measure, and $\mu(e)=\mu(e^*)$ for each Borel set e.

Recall that from the spectral theorem in the complex case we have $(\bar{A}^n)(\bar{A}^*)^m = =\int \lambda^n\bar{\lambda}^m d\bar{E}$, so letting $\lambda=r(\cos+i\sin\theta)$ and expanding this component-wise give $A^nA^{*m}x=\int r^{n+m}\cos(n-m)\theta dE_1 x-\int r^{n+m}\sin(n-m)\theta dE_2 x$ and $(A^nx, A^mx) = =\int r^{n+m}\cos(n-m)\theta d\mu=(f_n, f_m)$, where $f_n(r,\theta)=r^n(\cos n\theta+\sin n\theta)$. This follows since $f_n\cdot f_m = r^{n+m}(\cos(n-m)\theta+\sin(n+m)\theta)$, and $r^{n+m}\sin(n+m)\theta$ is an odd function in θ. So one has $(A^nA^{*m}x, A^kA^{*g}x) = (f_{n+g}, f_{k+m}) = \int r^{n+m}(\cos(n-m)\theta+ +\sin(n-m)\theta)r^{k+g}(\cos(k-g)\theta+\sin(k-g)\theta)d\mu$.

If one defines $\Phi(\Sigma a_{nm}A^nA^{*m}x) = \Sigma a_{nm}r^{n+m}(\cos(n-m)\theta+\sin(n-m)\theta)$ then Φ is well-defined and is an isometry from the linear span of the $A^nA^{*m}x$ into $L_2(\mu)$. Moreover, its range is dense in H since $\Phi(\frac{1}{2}(A^nA^{*m}+A^{*n}A^m)x) = r^{n+m}\cos(n-m)\theta$, $\Phi(\frac{1}{2}(A^nA^{*m}-A^{*n}A^m)x) = r^{n+m}\sin(n-m)\theta$, and the span of these functions is dense in $L_2(\mu)$. So Φ has a unique isometric extension of H onto L_2.

Recall the operator T defined in the Introduction. One can show by a straightforward calculation that $\Phi A\Phi^{-1}=T$ on the functions $r^{n+m}\cos(n-m)\theta$, $r^{n+m}\sin(n-m)\theta$, and hence for all of $L_2(\mu)$. Thus A is "orthogonally equivalent" to T.

Theorem 2. *If A is a bounded normal operator on the real Hilbert space H and if H contains a cyclic vector then there exists an $L_2(\mu)$ with $\mu(e)=\mu(e^*)$, such that A is orthogonally equivalent to T on $L_2(\mu)$* (see Introduction).

If there is no cyclic vector then apply Zorn's Lemma, see [1, pp. 910], to obtain $H=\oplus H_\alpha$ so that each H_α contains a cyclic vector x_α.

Theorem 3. *Every bounded normal operator A on the real Hilbert space H is* *orthogonally equivalent to an orthogonal sum $\oplus T_\alpha$ of operators on spaces $L_2(\mu_\alpha)$ of the type defined in the introduction.*

Remark 1. If one defines

$$E_1(x) = \begin{pmatrix} f_1 & 0 \\ 0 & f_1 \end{pmatrix}, \quad E_2(e) = \begin{pmatrix} 0 & -f_2 \\ f_2 & 0 \end{pmatrix},$$

where f_1 is the even part of χ_e, and f_2 is the odd part of χ_e then one can directly verify that (E_1, E_2) is a spectral pair, and for continuous f, $T(f) = \left(\int r\cos\theta dE_1\right)(f) - \left(\int r\sin\theta dE_2\right)(f)$, and so from uniqueness, (E_1, E_2) is the spectral pair for the operator T.

Remark 2. One could define a calculus for A by defining $f(A) = \int f_1 dE_1 - \int f_2 dE_2$, where $f = f_1 + if_2$ with f_1 even, f_2 odd, and both Borel measurable. The details are similar to [1, pp. 895—902].

Remark 3. As a corollary to Theorem 1 one has the self-adjoint case (3, pp. 269—276). The unbounded case follows from the bounded case just as in (3, pp. 313—320). Also, one could now write out the unitary and skew-symmetric cases from Theorem 1 and Theorem 3. Also, one could easily show that for compact normal operators H is the orthogonal direct sum of one and two dimensional invariant subspaces.

One could further use the above Theorem 1 in the skew-symmetric case and the methods found in (3, pp. 296—320) and (3, pp. 314—315) to obtain a spectral theorem for unbounded skew-symmetric operators in a real Hilbert space. Then using this theorem one could obtain Stone's theorem for real Hilbert space, see [2, pp. 38].

Added in proofs. The author has learned through private communication with Prof. TIN KIN WONG of Wayne State University that he has obtained some of the results of this paper by other methods. Also, Prof. Wong obtains the un bounded normal case by his methods. One could use the above methods and the unbounded self-adjoint and skew-symmetric cases to obtain the spectral theorem for unbounded normal operators.

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О некоторых инвариантных подпространствах диссипативных операторов экспоненциального типа

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Линейный ограниченный оператор A, действующий в сепарабельном гильбертовом пространстве $\mathfrak{H}$, будем относить к классу $\Lambda^{(\exp)}$, если: 1) A — диссипативен[1]); 2) A не имеет отличных от нуля точек спектра; 3) $(I-\lambda A)^{-1}$— функция экспоненциального типа. Тип роста функции $(I-\lambda A)^{-1}$ условимся обозначать через $\sigma(A)$. Если A удовлетворяет условиям 1), 2) и, кроме того, $\tau(A)=\operatorname{sp} A_I<\infty$ (соответственно $\dim A_I\mathfrak{H}=1$), то A будем относить к классу Λ_v (соответственно Λ_1).

Имеют место соотношения $\Lambda_1\subset\Lambda_v\subset\Lambda^{(\exp)}$. Для каждого оператора $A\in\Lambda^{(\exp)}$ выполняется неравенство $\sigma(A)\leqq 2\tau(A)$. Вполне несамосопряженные[2]) операторы класса Λ_1 одноклеточны и для них $\sigma(A)=2\tau(A)$ [1].

Согласно теореме Г. Э. Кисилевского [2] пространство $\mathfrak{H}$, в котором действует вполне несамосопряженный оператор $A\in\Lambda_v$, представимо в виде аппроксимативной суммы[3]) конечного или счетного числа инвариантных относительно A подпространств $\mathfrak{H}_j$, в каждом из которых индуцируется одноклеточный оператор A_j, причем числа $\sigma(A_j)$ определяются по оператору A однозначно. Для полного перенесения жордановой теории конечномерных операторов на класс Λ_v следует, очевидно, дополнить теорему Г. Э. Кисилевского решением обратной задачи, заключающейся в описании процесса конструирования любых операторов класса Λ_v из одноклеточных операторов того же

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[1]) Оператор A называется *диссипативным*, если $A_I=\dfrac{A-A^*}{2i}\geqq 0$.

[2]) Оператор A называется *вполне несамосопряженным*, если не существует инвариантного относительно A и A^* ненулевого подпространства, в котором индуцируется самосопряженный оператор.

[3]) Пространство $\mathfrak{H}$ называется *аппроксимативной суммой* своих подпространств $\mathfrak{H}_\gamma(\gamma\in\Gamma)$ если 1) $\tilde{\bigcup\limits_{\gamma\in\Gamma}}\mathfrak{H}_\gamma=\mathfrak{H}$; 2) $\left(\tilde{\bigcup\limits_{\gamma\in\Gamma_1}}\mathfrak{H}_\gamma\right)\cap\left(\tilde{\bigcup\limits_{\gamma\in\Gamma_2}}\mathfrak{H}_\gamma\right)=0$ для любых $\Gamma_1,\Gamma_2\subset\Gamma$, удовлетворяющих условиям $\Gamma_1\cup\Gamma_2=\Gamma$, $\Gamma_1\cap\Gamma_2=0$. Символом $\tilde{\bigcup\limits_{\gamma\in\Gamma}}\mathfrak{H}_\gamma$ обозначается замыканиелинейной оболочки подпространств $\mathfrak{G}_\gamma(\gamma\in\Gamma)$.

класса. О возникающих здесь затруднениях дает представление уже простейший случай, когда пространство $\mathfrak{H}$ представлено в виде аппроксимативной суммы двух своих подпространств $\mathfrak{H}_1$ и $\mathfrak{H}_2$. Дело в том, что если задать в них произвольно некоторые одноклеточные операторы $A_1 \in \Lambda_v$ и $A_2 \in \Lambda_v$, то этим в $\mathfrak{H}$ будет определен оператор A, который может оказаться неограниченным или недиссипативным.

В настоящей статье изучается взаимное расположение инвариантных подпространств операторов класса $\Lambda^{(\exp)}$, в которых индуцируются операторы класса Λ_1. В частности, решается вышеупомянутая обратная задача для тех операторов класса Λ_v, которые допускают распад на одноклеточные операторы класса Λ_1.

Мы будем ссылаться на следующие результаты [1].

Теорема 0. 1. *Если A — вполне несамосопряженный оператор класса $\Lambda^{(\exp)}$, действующий в $\mathfrak{H}$, и $\sigma(A)=0$, то $\mathfrak{H}=0$.*

Теорема 0. 2. *Если оператор A_0 индуцирован опертатором $A \in \Lambda^{(\exp)}$ в некотором инвариантном относительно A подпространстве, то $A_0 \in \Lambda^{(\exp)}$ и $\sigma(A_0) \leqq \sigma(A)$.*

Теорема 0. 3. *Пусть A — вполне несамосопряженный оператор класса $\Lambda^{(\exp)}$, действующий в пространстве $\mathfrak{H}$, и $A_\gamma (\gamma \in \Gamma)$ — операторы, индуцированные в некоторых инвариантных относительно A подпространствах $\mathfrak{H}_\gamma$. Если $\mathfrak{H} = \tilde{\bigcup\limits_{\gamma \in \Gamma}} \mathfrak{H}_\gamma$, то $\sigma(A) = \sup\limits_{\gamma \in \Gamma} \sigma(A_\gamma)$.*

Теорема 0. 4. *Пусть при условиях предыдущей теоремы подпространства $\mathfrak{G}_\gamma$ упорядочены по вложению. Если $A \in \Lambda_\gamma$, то $\inf\limits_{\gamma \in \Gamma} \sigma(A_\gamma) = \sigma(A_0)$, где оператор A_0 индуцирован в подпространстве $\mathfrak{H}_0 = \bigcap\limits_{\gamma \in \Gamma} \mathfrak{H}_\gamma$.*

1. Пусть $\mathfrak{G}$ — некоторое бесконечномерное сепарабльное гильбертово пространство. Построим гильбертово пространство $L^{(2)}_{\mathfrak{G}}(0, l)$ $(0 < l < \infty)$, состоящее из всех слабо измеримых вектор-функций $f(x)$ $(0 \leqq x \leqq l)$ со значениями в $\mathfrak{G}$, для которых $\|f\|^2 = \int\limits_0^l \|f(x)\|^2_{\mathfrak{G}} dx < \infty$. Скалярное произведение в $L^{(2)}_{\mathfrak{G}}(0, l)$ определяется формулой $(f, g) = \int\limits_0^l (f(x), g(x))_{\mathfrak{G}} dx$. Зададим в $L^{(2)}_{\mathfrak{G}}(0, l)$ оператор J, полагая

$$(Jf)(x) = 2i \int\limits_x^l f(y)\, dy,$$

и отнесем каждому вектору $g \in \mathfrak{G}$ векор-функцию $\hat{g} = \hat{g}(x) \equiv g$ $(0 \leqq x \leqq l)$. Легко видеть, что совокупность всех вектор-функций $\hat{g}(x)$ совпадает с областью значений оператора J_I. Оператор J вполне несамосопряжен и принадлежит классу $\Lambda^{(\exp)}$ [1].

Лемма 1. 1. *Если P — ортопроектор на инвариантное относительно J подпространство $L\subset L^{(2)}_{\mathfrak{G}}(0, l)$, то*

$$(1)\qquad \int_0^x ((f-Pf)(x-y), (Pg)(l-y))_{\mathfrak{G}}\,dy = 0 \qquad (0 \leqq x \leqq l;\ f(x), g(x)\in L^{(2)}_{\mathfrak{G}}(0, l)).$$

Доказательство. Поскольку

$$(J^{*n}h)(t) = \frac{(-2i)^n}{(n-1)!}\int_0^t s^{n-1}h(t-s)\,ds \qquad (h(x)\in L^{(2)}_{\mathfrak{G}}(0, l);\ n = 1, 2, \dots)$$

и

$$PJ^{*n}(I-P) = 0 \qquad (n = 1, 2, \dots),$$

то

$$\int_0^l s^{n-1}\int_s^l ((f-Pf)(t-s), (Pg)(t))_{\mathfrak{G}}\,dt\,ds = \int_0^l\int_0^t (s^{n-1}(f-Pf)(t-s), (Pg)(t))_{\mathfrak{G}}\,ds\,dt =$$

$$= \int_0^l \Big(\int_0^t s^{n-1}(f-Pf)(t-s)\,ds, (P)g(t)\Big)_{\mathfrak{G}}\,dt = \frac{(n-1)!}{(-2i)^n}\big(J^{*n}(I-P)f, Pg\big) = 0.$$

Следовательно,

$$(2)\qquad \int_s^l ((f-Pf)(t-s)\,(Pg)(t))_{\mathfrak{G}}\,dt = 0.$$

Равенство (1) вытекает из (2) при помощи замены переменных $s = l-x$, $t = l-y$. Лемма доказана.

Пусть $h\in\mathfrak{G}$ $(h\neq 0)$ и $\tau\in(0, l]$. Обозначим чрез $L(h, \tau)$ совокупность всех вектор-функций вида $\varphi(x)h$ $(0\leqq x\leqq l)$, где $\varphi(x)$ — произвольная скалярная функция с суммируемым квадратом модуля, равная нулю почти всюду на промежутке $[\tau; l]$. Очевидно, $L(h, \tau)$ представляет собой подпространство в $L^{(2)}_{\mathfrak{G}}(0, l)$, инвариантное относительно оператора J. Оператор, индуцированный в $L(h, \tau)$ будем обозначать через $J(h, \tau)$. Нетрудно проверить, что $J(h, \tau)\in\Lambda_1$. Область значений оператора $(J(h, \tau))_I$ натянута на вектор-функцию

$$h(\tau; x) = \begin{cases} h & (0\leqq x<\tau) \\ 0 & (\tau\leqq x\leqq l)\end{cases},$$

причем $(J(h, \tau))_I h(\tau; x) = \tau h(\tau; x)$.

Лемма 2. 1. *Каждое инвариантное относительно J подпространство $L\subset L^{(2)}_{\mathfrak{G}}(0, l)$, в котором индуцируется оператор класса Λ_1, совпадает с одним из подпространств $L(h, \tau)$.*

Доказательство. Пусть J_L — оператор, индуцированный в L, и $e(x)$ — орт, принадлежащий одномерному подпространству $(J_L)_I L$. Проекция на L каждой вектор-функции $\hat{f}(x)$ коллинеарна $e(x)$. Подставляя в (1) вместо $f(x)$ и $g(x)$ соответственно $\hat{f}(x)$ и $e(x)$, получим

$$\begin{aligned}\int_0^x (\hat{f}, e(l-y))_{\mathfrak{G}}\,dy &= \int_0^x ((\hat{f}, e)e(x-y), e(l-y))_{\mathfrak{G}}\,dy,\\ \int_0^x e(l-y)\,dy &= \int_0^x (e(l-y), e(x-y))_{\mathfrak{G}}\,dy \int_0^l e(y)\,dy.\end{aligned}\tag{3}$$

Дифференцируя обе части последнего соотноения и полагая $h=\int_0^l e(y)dy$, придем к равенству $e(x)=\varphi(x)h$, где $\varphi(x)$ — скалярная функция. Из (3) следует теперь, что

$$\int_0^x \left(1-\|h\|_{\mathfrak{G}}^2\overline{\varphi(x-y)}\right)\varphi(l-y)\,dy = 0.$$

По теореме Титчмарша о свертке [3] существует такое число $\tau\in[0, l]$, что

$$\varphi(x)=\begin{cases}\|h\|_{\mathfrak{G}}^{-2} & (0\leqq x<\tau),\\ 0 & (\tau\leqq x\leqq l).\end{cases}$$

Остается заметить, что L представляет собой замыкание линейной оболочки вектор-функций вида $J^n(\varphi(x)h)$ $(n=0, 1, \dots)$.

Лемма 3. 1. *Если $0<\tau\leqq\tau_0\leqq l$, то то проекция вектор-функции $h_0(\tau_0; x)$ на подпространство $L(h, \tau)$ коллинеарна вектор-функции $h(\tau; x)$.*

Доказательство. Пусть P — ортопроектор на $L(h, \tau)$. Тогда

$$Ph_0(\tau_0; x) = \psi(x)h \quad (\psi(x)\in L^{(2)}(0, l);\ \psi(x) = 0\ (\tau\leqq x\leqq l))$$

и, следовательно,

$$\int_0^\tau (h_0(\tau_0; x), \varphi(x)h)_{\mathfrak{G}}\,dx = \int_0^\tau (\psi(x)h, \varphi(x)h)_{\mathfrak{G}}\,dx \qquad (\varphi(x)\in L^{(2)}(0, \tau)).$$

Таким образом,

$$(h_0(\tau_0; x), h)_{\mathfrak{G}} = \psi(x)(h, h)_{\mathfrak{G}}, \quad \psi(x)\equiv\frac{(h_0, h)_{\mathfrak{G}}}{(h, h)_{\mathfrak{G}}} \qquad (0\leqq x<\tau).$$

Теорема 1. 1. *Пусть $h_j\in\mathfrak{G}$ $(j=1, 2, \dots, n)$ и $\tau_j\in(0, l]$ $(j=1, 2, \dots, n)$. Тогда либо*

$$\dot{\bigcup_{j=1}^n} L(h_j, \tau_j) = L(h_1, \tau_1)\dot{+}L(h_2, \tau_2)\dot{+}\dots\dot{+}L(h_n, \tau_n)\tag{4}$$

и вместе с тем

(5) $$\tilde{\bigcup_{j=1}^{n}} L(h_j, \tau_j) = \bigoplus_{j=1}^{n} L(e_j, \tau_j),$$

где $\{e_j\}_1^n$ *— некоторая ортонормированная система, либо существует число* $j_0 (1 \leqq j_0 \leqq n)$ *такое, что*

$$L(h_{j_0}, \tau_{j_0}) \subset \tilde{\bigcup_{j \neq j_0}} L(h_j, \tau_j).$$

Доказательство. Не нарушая общности, можно считать, что

(6) $$\tau_1 \geqq \tau_2 \geqq \ldots \geqq \tau_n.$$

Пусть векторы h_j $(j=1, 2, \ldots, n)$ линейно независимы. Тогда, образуя ортонормированную систему

$$e_j = \alpha_{j1} h_1 + \alpha_{j2} h_2 + \ldots + \alpha_{jj} h_j \qquad (j = 1, 2, \ldots, n)$$

и пользуясь соотношениями (6), получим

(7) $$L(e_j, \tau_j) \subset \tilde{\bigcup_{k=1}^{j}} L(h_k, \tau_k).$$

При этом

$$h_j = \beta_{j1} e_1 + \beta_{j2} e_2 + \ldots + \beta_{jj} e_j \qquad (j = 1, 2, \ldots, n)$$

и, следовательно,

(8) $$L(h_j, \tau_j) \subset \bigoplus_{k=1}^{j} L(e_k, \tau_k).$$

Из (7) и (8) вытекает равенство (5). Поскольку каждая вектор-функция, содержащаяся в правой части равенства (5), однозначно представима в виде суммы n слагаемых, принадлежащих соответственно подпространствам $L(h_j, \tau_j)$ $(j=1, 2, \ldots, n)$, то верно и равенство (4).

В случае, когда вектор h_{k_0} является линейной комбинацией векторов $h_1, h_2, \ldots, h_{k_0-1}$, имеем

$$L(h_{k_0}, \tau_{k_0}) \subset \tilde{\bigcup_{k=1}^{k_0-1}} L(h_k, \tau_k).$$

Теорема 2.1. *Пусть* $h^{(\gamma)} \in \mathfrak{G}$ *и* $\tau^{(\gamma)} \in (0, l]$ $(\gamma \in \Gamma)$. *Если в подпространстве* $L = \tilde{\bigcup_{\gamma \in \Gamma}} L(h^{(\gamma)}, \tau^{(\gamma)})$ *индуцируется оператор* A *класса* Λ_v, *то*

(9) $$L = \bigoplus_{j=1}^{\omega} L(e_j, \tau_j) \qquad (\omega \leqq \infty),$$

где $\{e_j\}_1^{\omega}$ *— некоторая ортогональная система, а числа* τ_j *удовлетворяют соотношениям*

(10) $$\tau_1 = \tau_2 = \ldots = \tau_{p_1} > \tau_{p_1+1} = \tau_{p_1+2} = \ldots = \tau_{p_2} > \tau_{p_2+1} = \ldots .$$

Представление (9) *является единственным в том смысле, что если*

$$L = \bigoplus_{j=1}^{\omega'} L(e_j', \tau_j') \qquad (\omega' \leqq \infty), \tag{11}$$

где $\{e_j'\}_1^{\omega'}$ *— новая ортонормированная система, и*

$$\tau_1' = \tau_2' = \ldots = \tau_{p_1'}' > \tau_{p_1'+1}' = \tau_{p_1'+2}' = \ldots = \tau_{p_2'}' > \tau_{p_2'+1}' = \ldots, \tag{12}$$

то $\omega = \omega'$, $\tau_j = \tau_j'$ $(j = 1, 1, 2, \ldots, \omega)$ *и*

$$\bigoplus_{j=p\ +1}^{p_{k+1}} L(e_j, \tau_j) = \bigoplus_{j=p_k+1}^{p_{k+1}} L(e_j', \tau_j) \qquad (k = 0, 1, \ldots; \ p_0 = 0). \tag{13}$$

Доказательство. Ввиду теоремы 0. 3 $\tau_1 = \sup_{\gamma \in \Gamma} \tau^{(\gamma)} < \infty$. Покажем, что существует вектор $e_1 \in \mathfrak{G}$ $(\|e_1\|_{\mathfrak{G}} = 1)$ такой, что $L(e_1, \tau_1) \subset L$. Это утверждение очевидно, если τ_1 совпадает с одном из чисел $\tau^{(\gamma)}$. Если же все $\tau^{(\gamma)}$ отличны от τ_1, то в силу теоремы 1. 1 и условия $A \in \Lambda_v$ существует конечномерное подпространство $\mathfrak{G}_0 \subset \mathfrak{G}$ и такая последовательность $L(h^{(\gamma_j)}, \tau^{(\gamma_j)})$, что 1) $\tau^{(\gamma_1)} \leqq \tau^{(\gamma_2)} \leqq \ldots$, 2) $\tau^{(\gamma_j)} \to \tau_1$, 3) векторы $h^{(\gamma_{kn+1})}, h^{(\gamma_{kn+2})}, \ldots, h^{(\gamma_{kn+n})}$ при каждом целом неотрицательном k образуют базис в $\mathfrak{G}_0$. Отсюда вытекает, что при любом $e_1 \in \mathfrak{G}_0$ $(\|e_1\|_{\mathfrak{G}} = 1)$ подпространства $L(e_1, \tau^{(\gamma_{kn+1})})$ $(k = 0, 1, \ldots)$ приадлежат L. Следовательно $L(e_1, \tau_1) \subset L$.

По теореме 1. 1 линейная оболочка подпространств $L(e_1, \tau_1)$ и $L(h^{(\gamma_0)}, \tau^{(\gamma_0)})$ $(\gamma_0 \in \Gamma)$ либо совпадает с $L(e_1, \tau_1)$, либо представима в виде $L(e_1, \tau_1) \oplus \oplus L(g^{(\gamma_0)}, \tau^{(\gamma_0)})$. Поэтому $L = L(e_1, \tau_1) + L_1$, где L_1 — замыкание линейной оболочки некоторых подпространств $L(g^{(\delta)}, \tau^{(\delta)})$ $(\delta \in \Delta)$ таких, что $\tau_2 = \sup_{\delta \in \Delta} \tau^{(\delta)} \leqq \leqq \tau_1$. Представляя аналогично L_1 в виде $L(e_2, \tau_2) + L_2$ и продолжая этот процесс, получим одно из следующих соотношений:

$$L = L(e_1, \tau_1) \oplus L(e_2, \tau_2) \oplus \ldots \oplus L(e_\omega, \tau_\omega),$$

$$L = \bigoplus_{j=1}^{\infty} L(e_j, \tau_j) \oplus L'. \tag{14}$$

Рассмотрим равенство (14) и обозначим через A_k оператор, индуцированный в подпространстве

$$L_k = L \ominus \left[\bigoplus_{j=1}^{k} L(_j, \tau_j) \right].$$

Так как

$$\sum_{j=1}^{\infty} \tau_j < \infty, \quad \sigma(A_k) = 2\tau_k \to 0 \tag{15}$$

и $L' = \bigcap_{k=1}^{\infty} L_k$, то в силу теорем 0. 4 и 0. 1 $L' = 0$.

Мы показали, что подпространство L представимо в виде (9). При этом, ввиду первого из соотношений (15), выполняется условие (10). Если одновременно имеет место представление (11) и для чисел τ'_j выполняются условия (12), то по теореме Г. Э. Кисичевского [2] $\omega=\omega'$ и $\tau_j=\tau'_j$ $(j=1, 2, \ldots, \omega)$. Согласно лемме 3. 1

$$e'_j(\tau_j;\, x) = \sum_{k=1}^{\omega} c_{jk} e_k(\tau;x), \qquad e_j(\tau_j;x) = \sum_{k=1}^{\omega} d_{jk} e'_k(\tau_k;x) \quad (j=1,2,\ldots,p_1;0\leqq x\leqq l),$$

откуда легко следует, что

$$e'_j = \sum_{k=1}^{p_1} c_{jk} e_k, \quad e_j = \sum_{k=1}^{p_1} d_{jk} e'_k \qquad (j = 1, 2, \ldots, p_1).$$

Применяя теорему 1. 1, найдем, что

$$\bigoplus_{j=1}^{p_1} L(e_j, \tau_j) = \bigoplus_{j=1}^{p_1} L(e'_j, \tau_j).$$

Остальные равенства (13) доказываются аналогично.

2. Пусть в пространстве $\mathfrak{H}$ задан вполне несамосопряженный оператор A класса $\Lambda^{(\exp)}$. В пространстве $L^{(2)}_{\mathfrak{G}}(0, l)$ $\big(2l=\sigma(A)\big)$ существует инвариантное относительно J подпространство L такое, что индуцированный в L оператор J_L удовлетворяет условию $UA=J_LU$, где U — некоторое изометрическое отображение $\mathfrak{H}$ на L [4]. Этот результат позволяет переформулировать теоремы 1. 1 и 2. 1 следующим образом.

Теорема 1. 2. *Если A — вполне несамосопряженный оператор класса $\Lambda^{(\exp)}$ и $\mathfrak{H}_j$ $(j=1, 2, \ldots, n)$ — инвариантные относительно A подпространства, в которых индуцируются операторы класса Λ_1, то либо*

$$\tilde{\bigcup_{j=1}^{n}} \mathfrak{H}_j = \mathfrak{H}_1 \dot{+} \mathfrak{H}_2 \dot{+} \cdots \dot{+} \mathfrak{H}_n,$$

либо существует число j_0 $(1\leqq j_0\leqq n)$ такое, что $\mathfrak{H}_{j_0}\subset \tilde{\bigcup}_{j\neq j_0} \mathfrak{H}_j$.

Теорема 2. 2. *Пусть A — вполне несамосопряженный оператор класса Λ_v и $\mathfrak{H}^{(\gamma)}$ $(\gamma\in\Gamma)$ — инвариантные относительно A подпространства, в которых индуцируются операторы класса Λ_1. Тогда $\tilde{\bigcup}_{\gamma\in\Gamma} \mathfrak{H}^{(\gamma)} = \bigoplus_{j=1}^{\omega} \mathfrak{H}_j$ $(\omega\leqq\infty)$, где подпространства $\mathfrak{H}_j$ инвариантны относительно A, а индуцированные в них опе-*

раторы A_j принадлежат Λ_1 и удовлетворяют условиям

$$\tau(A_1)=\tau(A_2)=\ldots=\tau(A_{p_1})>\tau(A_{p_1+1})=\tau(A_{p_1+2})=\ldots=\tau(A_{p_2})>\tau(A_{p_2+1})=\ldots\ .\ ^{4})$$

Если, кроме того, $\tilde{\bigcup\limits_{\gamma\in\Gamma}}\mathfrak{H}^{(\gamma)}=\bigoplus\limits_{j=1}^{\omega'}\mathfrak{H}'_j\,(\omega'\leqq\infty)$, *где подпространства* $\mathfrak{H}'_j$ *также инвариантны относительно* A, *а индуцированные в них операторы* A'_j *принадлежат* Λ_1 *и удовлетворяют соотношениям*

$$\tau(A'_1)=\tau(A'_2)=\ldots=\tau(A'_{p'_1})>\tau(A'_{p'_1+1})=\tau(A'_{p'_1+2})=\ldots=\tau(A'_{p'_2})>\tau(A'_{p'_2+1})=\ldots,$$

то $\omega=\omega'$, $\tau(A_j)=\tau(A'_j)$ $(j=1, 2, \ldots, \omega)$ *и*

$$\bigoplus_{j=\bar{p}_k+1}^{p_{k+1}}\mathfrak{H}_j=\bigoplus_{j=\bar{p}_k+1}^{p_{k+1}}\mathfrak{H}'_j\qquad(k=0, 1, \ldots;\ p_0=0).$$

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[4]) Ортогональный распад на операторы класса Λ_1 был получен Г. Э. Кисилевским [5] для произвольного вполне несамосопряженного оператора $A\in\Lambda_v$ с n-мерной мнимой компонентой, удовлетворяющего условию $\sigma(A)=\dfrac{2\tau(A)}{n}$.

A metric characterization of homogeneous Riemannian manifolds

By J. SZENTHE in Szeged

Let M be a Riemannian manifold and $\varrho(x, y)$ the infimum of the length of those piecewise C^1-curves which join x, y in M. As well-known ϱ is a distance function on M and the thus *induced metric space* $[M, \varrho]$ is so closely related to the Riemannian manifold that a considerable number of theorems about it can be formulated and proved merely in terms of $[M, \varrho]$. This circumstance can be regarded as the starting point of the theories of H. BUSEMANN and W. RINOW where a metric space is the basic concept and some fundamental properties common to all metric spaces induced by Riemannian or Finsler manifolds are being postulated. Although these theories go beyond the scope of the standard one, e.g. as to differentiability conditions, their exact relation to it is not sufficiently clarified yet. In other words no adequate necessary and sufficient conditions are known which imply that a metric space should be induced by a Riemannian manifold. A partial solution of this problem is presented below, i.e. necessary and sufficient conditions are given for the case of metric spaces induced by homogeneous Riemannian manifolds.

1. Basic concepts and the main result

Some well-known fundamental facts concerning metric spaces induced by C^∞-Riemannian manifolds are summarized here. (For a detailed presentation see [5].)

A metric space is said to be *finitely compact* if any bounded infinite subset has a point of accumulation in it. Metric spaces induced by complete Riemannian manifolds are finitely compact. A locally distance preserving map of the real line into a metric space is called a *geodesic*. The geodesics of a Riemannian manifold which are parametrized in terms of arc length and geodesics of its induced metric space are the same. If a, b, c are distinct points of a metric space $[R, \varrho]$ and $\varrho(a, c)+$ $+\varrho(c, b) = \varrho(a, b)$, then it is said that *c lies between a* and b, in notation: acb. If $A \subset R$ and to any two different points a, b of A there is a $c \in A$ with acb, then A is said to be *convex*. The induced space of a complete Riemannian manifold is convex. A distance preserving map of a compact interval of the real line into a metric space is called a *segment*. If $[R, \varrho]$ is a finitely compact and convex metric space then any two points can be joined by a segment in it. The segments are said to be *locally prolongable* in a finitely compact convex metric space $[R, \varrho]$ if to any $p \in R$ there is such a $\delta_p > 0$ that to any two distinct points a, b in $B(p, \delta_p) = \{x : \varrho(x, p) < \delta_p\}$ there is a $c \in R$ with abc. It is said that the *prolongation of segments is unique* in $[R, \varrho]$ if $x, y, z', z'' \in R$, xyz', xyz' and $\varrho(x, z') = \varrho(x, z'')$ imply $z' = z''$. The above terminology is justified by the fact that the segments of a finitely compact convex metric space are uniquely extendable to geodesics if the preceding two conditions hold. The closed subset $A \subset R$ is called *strictly convex* if it is convex and $a, b, c \in A$, acb imply that $c \in \operatorname{int} A$. The metric space $[R, \varrho]$ is called *regular* if to any $p \in R$ there are such $\varkappa_p, \lambda_p > 0$ that the closed balls $\overline{B(x, \xi)}$ are strictly convex if $x \in B(p, \varkappa_p)$ and $0 < \xi \leqq \lambda_p$. Riemannian manifolds induce regular metric spaces.

The induced metric space of a Finsler manifold can be defined analogously and the above facts generalize to their case as well; see [9]. A connection with the induced metric space peculiar to Riemannian manifolds can be expressed in terms of the metric angle concept. Let a, b, c be points of a metric space $[R, \varrho]$ then there are points A, B, C of the euclidean plane with $\varrho(a, b) = \overline{AB}$, $\varrho(b, c) = \overline{BC}$, $\varrho(c, a) = \overline{CA}$. If $a \neq b, c$, then by the metric angle $\gamma(a; b, c)$ of the triple $\{a, b, c\}$ at a the measure of $\sphericalangle BAC$ is meant. Let $\varphi, \psi : [0, \alpha] \to R$ be continuous curves with $\varphi(0) = \psi(0) = x$ and with such a $0 < \delta \leqq \alpha$ that $\varphi(\tau), \psi(\tau) \neq x$ for $0 < \tau \leqq \delta$. If $\gamma(\varphi, \psi) = \lim\limits_{\tau' \tau'' \to 0} \gamma(x; \varphi(\tau'), \psi(\tau''))$ exists, then this value is called the *metric angle* of φ and ψ at x. If φ, ψ are differentiable curves of a Riemannian manifold then considered as curves of the induced metric space they have a metric angle which is equal to the one which they have as curves of the Riemannian manifold; see [7].

An isometric transformation of a Riemannian manifold is obviously a distance preserving transformation of its induced metric space. The converse of this assertion is a theorem due to S. B. Myers and N. Steenrod (see [6]).

Let $\Phi : R^1 \times S \to S$ be a continuous 1-parameter group of transformations of the topological space S, then the continuous curve $\varphi : R^1 \to S$ defined by $\varphi(\tau) = \Phi(\tau, x)$, $\tau \in R^1$ is called the *orbit of Φ starting at $x \in S$*.

The main result of this paper is the following

Theorem 1. *Let $\Gamma : G \times R \to R$ be an effective and transitive transformation group, where R has a distance function ϱ such that the elements of G are distance preserving transformations of $[R, \varrho]$. Assume that*

1. *$[R, \varrho]$ is finitely compact,*
2. *$[R, \varrho]$ is convex,*

3. *the segments are locally prolongable in* $[R, \varrho]$,
4. *the prolongation of segments is unique*,
5. $[R, \varrho]$ *is regular*,
6. *the orbits of* 1*-parameter groups of distance preserving transformations are rectifiable in* $[R, \varrho]$,
7. *if two such orbits have a point in common then they have a metric angle there.*

Then G *with the compact-open tolpogy is a topological group and* Γ *is a continuous transformation group. The identity component* G_0 *of* G *is a Lie group and* R *has a unique differentiable manifold structure such that* $\Gamma_0 : G_0 \times R \to R$, *the restriction of* Γ, *is a transitive differentiable transformation group. There is a unique Riemannian manifold structure on* R *which has* $[R, \varrho]$ *as its induced metric space.*

Conditions 1—4 have been introduced by H. BUSEMANN [1] as the starting point for his theory of G-spaces.

The proof of the above theorem is carried out in two steps: first a differentiable structure is introduced on R, secondly a Riemannian structure. These two steps are summarized in Theorem 2 and 3. Theorem 1 is a direct consequence of these two theorems.

Conditions 1—7 of Theorem 1 will be generally assumed to hold in what follows. Differentiability will mean C^∞, unless it is not explicitely otherwise stated, although in some cases obviously less would suffice or more could be stated.

2. The introduction of the differentiable structure

The initial step in introducing the differentiable structure of R is the definition of an appropriate topology in the group of distance preserving transformations. This can be done by an obvious application of standard methods (see [5]) by proving

Lemma 2.1. *Let* $[R, \varrho]$ *be a finitely compact metric space and* $\Gamma : G \times R \to R$ *an effective transformation group where the elements of* G *are distance preserving transformations of* $[R, \varrho]$, *then with the compact-open topology* G *is a* σ*-compact group and* Γ *a toplogical transformation group.*

The next step is to show that the identity component G_0 of G is a Lie group. Owing to a theorem of A. GLEASON and H. YAMABE (see [3], [10]) it suffices to prove that G has no small subgroups. But this is asserted in the following lemma which has been proved already elsewhere (see [8]):

Lemma 2.2. *Let* $\Gamma : G \times R \to R$ *be an effective transformation group where* R *has a distance function* ϱ *such that* $[R, \varrho]$ *is a finitely compact convex and regular* *metric space in which segments are locally and uniquely prolongable and the elements of G are distance preserving transformations of* $[R, \varrho]$. *If G is taken with the compact-open topology then it has no small subgroups.*

The following facts are obvious consequences of well-known theorems. For any $x \in R$ the corresponding subgroup of stability $H_x \subset G$ is compact. Since Γ is transitive the elements of G which carry x into $y \in R$ form a subset $\Psi_x(y)$ of G which is a left coset of H_x, and if the left coset space G/H_x is endowed with the quotient topology then the map $\Psi_x: R \to G/H_x$ thus defined is a homeomorphism. Let $\Pi_x: G \to G/H_x$ be the natural projection then $\Pi(G_0)$ is a component of G/H_x. Since R is connected and homeomorphic to G/H_x the identity component G_0 is transitive on R. If $H_{0_x} = H_x \cap G_0$ then since G_0 is a Lie group the left coset space can be endowed with such a differentiable structure that the operation of G_0 on G_0/H_{0_x} by left translations is differentiable. Taking into account the homeomorphism $\Psi_{0_x}: R \to G_0/H_{0_x}$ defined analogously to Ψ_x, the above assertions yield

Theorem 2. *Let* $\Gamma: G \times R \to R$ *be an effective and transitive transformation group and R have a distance function* ϱ *such that* $[R, \varrho]$ *is a finitely compact convex and regular metric space in which the segments are locally and uniquely prolongable and the elements of G are distance preserving transformation of* $[R, \varrho]$. *If G is taken with the compact-open topology then its identity component* G_0 *is a Lie group and R can be endowed with such a differentiable structure that* $\Gamma_0: G_0 \times R \to R$ *the restriction of* Γ *to* $G_0 \times R$ *is a differentiable transformation group.*

For the sake of some of the subsequent and later arguments the main steps in the construction of the differentiable structure of R are summed up here. (For a detailed presentation see [4].) The tangent space $T_\varepsilon H_{0x}$ of H_{0x} at the identity ε is a subspace of $T_\varepsilon G_0$. Let M be a subspace of $T_\varepsilon G_0$ complementary to $T_\varepsilon H_{0x}$. A neighborhood of $O_x \in T_\varepsilon G_0$ is mapped diffeomorphically onto a neighborhood of ε by $\exp_\varepsilon: T_\varepsilon G_0 \varepsilon \to G_0$ and a neighborhood V of O_x in M is mapped homeomorphically onto a neighborhood U of H_{0x} in G_0/H_{0x} by $\Pi_x \circ \exp_\varepsilon: M \to G_0/H_{0x}$. Let $\bar{\varkappa}_x^{-1}$ be the restriction of $\Pi_x \circ \exp_\varepsilon$ to V, since M can be identified with R^m where $m = \dim M$, a coordinate system $\bar{\varkappa}_x: U \to R^m$ of G_0/H_{0x} is obtained. If $\alpha \in G_0$ the left translation $L_\alpha: G_0 \to G_0$ defines a homeomorphism $\bar{L}_\alpha: G_0/H_{0x} \to G_0/H_{0x}$ and $\bar{\varkappa}_x \circ \bar{L}_\alpha$ is a coordinate system on a neighborhood of $\alpha^{-1}H_{0x}$. Thus a differentiable atlas $\{\bar{\varkappa}_x \circ \bar{L}_\alpha : \alpha \in G_0\}$ of G_0/H_{0x} is constructed and this defines a differentiable structure which does not depend on the particular choice of M. For any $z \in R$ the analogously defined differentiable manifold G_0/H_{0z} is diffeomorphic to G_0/H_{0x}. Therefore the homeomorphism $\Psi_{0x}: R \to G_0/H_{0x}$ defines a differentiable structure of R which does not depend on x. The coordinate system $\varkappa_x = \bar{\varkappa}_x \circ \Psi_{0x}^{-1}: U_x \to R^m$ of R will be called a *canonical coordinate system of the first kind at x*.

If $\gamma: R^1 \to G_0$ is a 1-parameter group and $x \in R$ then the differentiable curve $\varphi: R^1 \to R$ defined by $\varphi(\tau) = \gamma(\tau)(x)$, $\tau \in R^1$ is called the orbit of γ starting at x. Let $\varkappa_x: U_x \to R^m$ be a canonical coordinate system of the first kind at x and d the distance function of R^m. If $v \in M$ has length equal to 1 with respect to d and γ is the 1-parameter group defined by $\gamma_*(0) = v$ then the orbit φ of γ starting at x will be called a *fundamental orbit* of the coordinate system $\varkappa_x$. If $z \in U_x$ and $z \neq x$ then there is a unique fundamental orbit φ of $\varkappa_x$ with $\varphi(\tau) = z$ where $\tau = d(\varkappa_x(z), \varkappa_x(x))$.

Let $\varkappa': U' \to R^m$, $\varkappa'': U'' \to R^m$ be coordinate systems of R with $U' \cap U'' \neq \emptyset$ and $\|\alpha_{ij}(u)\|_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$ the Jacobian of the map $\varkappa' \circ \varkappa''^{-1}: \varkappa'(U') \cap \varkappa''(U'') \to R^m$ at $\varkappa''(u)$ for $u \in U' \cap U''$. Let $\lambda(\varkappa', \varkappa'')$ be defined by

$$\lambda(\varkappa', \varkappa'') = \sqrt{(2m-1)m} \cdot \sup \{|\alpha_{ij}(u)| : u \in U' \cap U'', \ i, j = 1, \dots, m\}.$$

If $v \in T_u R$ and $(v'^1, \dots, v'^m), (v''^1, \dots, v''^m)$ are its coordinates in the coordinate systems $\varkappa', \varkappa''$ then obviously

$$\left[\sum_{i=1}^{m} (v'^i)^2\right]^{1/2} \leqq \lambda(\varkappa', \varkappa'') \left[\sum_{i=1}^{m} (v''^i)^2\right]^{1/2}.$$

The following lemma will prove useful in later arguments.

Lemma 2.3. *Any $x \in R$ has a compact neighborhood W such that to every $z \in W$ there is a canonical coordinate sytem of the first kind $\varkappa_z: U_z \to R^n$ at z with the following properties:*

1. *$W \subset U_z$ for $z \in W$;*
2. *there is a bound C with $\lambda(\varkappa_z, \varkappa_x) \leqq C$ for $z \in W$;*
3. *if $S(z) \subset T_z R$ is the set of vectors which are tangent to a fundamental orbit of $\varkappa_z$ then $\bigcup \{S(z): z \in W\}$ is a compact subset of TR.*

Proof. Let $M \subset T_\varepsilon G_0$ a subspace complementary to $T_\varepsilon H_{0x}$ be identified with R^m and $\varkappa_x: U_x \to R^m$ the corresponding canonical coordinate system of the first kind at x. If $z \in R$ then $H_{0z} = \alpha H_{0x} \alpha^{-1}$ for any $\alpha \in G_0$ with $z = \alpha(x)$, therefore $T_\varepsilon H_{0z} = L_{\alpha *} R_{\alpha^{-1} *}(T_\varepsilon H_{0x}) = \mathrm{ad}\, \alpha_*(T_\varepsilon H_{0x})$. This implies the existence of a neighborhood W' of x such that M is complementary to $T_\varepsilon H_{0z}$ for $z \in W'$. Let $\varkappa_z: U_z \to R^m$ be the canonical coordinate system of the first kind at z defined by M for $z \in W'$. If $y \in U_x$ then there is a ξ in the corresponding neighborhood of ε such that $y = \xi(x)$ and $\varkappa_x(y) = \exp_\varepsilon^{-1}(\xi H_{0x} \cap \tilde{M})$ where $\tilde{M} = \exp_\varepsilon(M)$. There is such a neighborhood W'' of x and $\overline{W}$ of ε that $\xi H_{0x} \cap \tilde{M}\alpha$ is a single point and $\exp_\varepsilon^{-1}(\xi H_{0x} \cap \tilde{M}\alpha)$ defines a coordinate system of R on the neighborhood W'' for $\alpha \in \overline{W}$. There is a neighborhood W''' of x such that for $z \in W'''$ there is a unique $\alpha \in \overline{W}$ with $z = \alpha(x)$ and $\alpha = \alpha H_{0x} \cap \tilde{M}$. Then by $\exp_\varepsilon^{-1}((\xi H_{0x} \cap \tilde{M}\alpha)\alpha^{-1})$ a coordinate system of R is defined on W'' for $z \in W'''$. But $\xi H_{0x} = \xi \alpha^{-1} H_{0z} \alpha = \eta H_{0z} \alpha$ with $\eta = \xi \alpha^{-1}$ and $y = \eta(z)$, therefore

$\exp^{-1}((\xi H_{0x}\cap\tilde{M}\alpha)\alpha^{-1})=\exp_\varepsilon^{-1}(\eta H_{0z}\cap\tilde{M})=\varkappa_z(y)$. Let W be a compact neighborhood of x with $W\subset W'\cap W''\cap W'''$. Then $W\subset U_z$ for $z\in W$ and the existence of the bound C follows from the differentiability of the coordinate systems and from the fact that α depends continuously on z, with a possible restriction of U_z to a compact neighborhood $U_z'\subset W$. Since $S(z)$, $z\in W$ is compact $\bigcup\{S(z):z\in W\}$ is compact as well.

A field of canonical coordinate systems of the first kind $\varkappa_z:U_z\to R^m$, $z\in W$ defined according the preceding proof will be called *normal.*

Let $M\subset T_\varepsilon G_0$ be a subspace complementary to $T_\varepsilon H_{0x}$ and $\{w_1,\dots,w_m\}\subset M$ a base of M. Then $v=\sum_{i=1}^{m}\alpha^i v_i$ is unique for $v\in M$ and by $\sigma(v)=\exp(\alpha^1 v_1)\dots\exp(\alpha^m v_m)$ a map $\sigma:M\to G_0$ is defined. With methods similar to those applied at the definition of canonical coordinates of the first kind (see [4]) it can be shown that $\bar{\varkappa}^{-1}=\Pi_x\circ\sigma$ maps diffeomorphically a neighborhood of O_ε in M onto a neighborhood of H_{0x} in G_0/H_{0x}. Thus $\varkappa=\bar{\varkappa}\circ\psi_{0x}:U\to R^m$ is a coordinate system of R which will be called a canonical *coordinate system of the second kind at* x. The proof of the following lemma is obvious.

Lemma 2.4. *Let $\{v_1,\dots,v_m\}$ be a base of T_xR then there are 1-parameter groups γ_i of G_0 with orbits φ_i starting at x and a canonical coordinate system of the second kind $\varkappa:U\to R^m$ at x such that $\varphi_{i*}(0)=v_i$ for $i=1,\dots,m$ and $z=\gamma_1(z^1)\circ\dots\circ\gamma_m(z^m)(x)$ for $z\in U$ with $\varkappa(z)=(z^1,\dots,z^m)$.*

3. The introduction of the Riemannian metric

Let $\varphi:R^1\to R$ be the orbit of the 1-parameter group $\gamma:R^1\to G_0$ starting at $x\in R$, then in consequence of the fact that φ is rectifiable $\gamma^*(x)=\lim\limits_{\tau\to 0}\dfrac{\varrho(\varphi(\tau),\varphi(0))}{|\tau|}$ exists. This defines a function $\gamma^*:R\to R^1$ which is constant on the orbits of γ, and it will be called the *velocity function* of γ. The value φ^* of γ^* on the orbit φ will be called the *velocity of the orbit.* An orbit is constant obviously if and only if its velocity is zero.

Lemma 3.1. *The velocity function γ^* of a 1-parameter group γ: $R^1\to G_0$ is continuous.*

Proof. Let φ be the orbit of γ starting at $x\in R$ and define $f_n:R\to R^1$ for $n=1,2,\dots$ by $f_n(x)=2^n\varrho(\varphi(1/2^n),\varphi(0))$. The functions f_n are continuous, $f_{n+1}(x)\geqq f_n(x)$ and $\gamma^*(x)=\lim\limits_{n\to\infty}f_n(x)$ hold for every $x\in R$. These imply the assertion.

A closer relation of the distance function ϱ and the differentiable structure of R is expressed by

Lemma 3.2. *If $\varkappa: U \to R^m$ is a coordinate system of R at x and d the distance function of R^m then there exist a neighborhood $V \subset U$ of x and a $\delta > 0$ such that $d(\varkappa(a), \varkappa(b)) \geqq \delta \cdot \varrho(a, b)$ if $a, b \in V$.*

Proof. Let $\bar{\varkappa}: U \to R^m$ be a canonical coordinate system of the second kind of R at x. There is a $\tilde{\delta} > 0$ such that $d(\varkappa(a), \varkappa(b)) \geqq \tilde{\delta} d(\bar{\varkappa}(a), \bar{\varkappa}(b))$ for $a, b \in U \cap \bar{U}$. If $\gamma_1, \ldots, \gamma_m$ are the 1-parameter groups which define $\bar{\varkappa}$, then by the preceding lemma there are a neighborhood $V \subset U \cap \bar{U}$ of x and a K such that $\gamma_1^*(z), \ldots, \gamma_m^*(z) \leqq K$ for $z \in V$. If $\bar{\varkappa}(a) = (\alpha^1, \ldots, \alpha^m)$, $\bar{\varkappa}(b) = (\beta^1, \ldots, \beta^m)$ for $a, b \in V$ then

$$\varrho(a, b) = \varrho\big(\gamma_1(\alpha^1) \circ \ldots \circ \gamma_m(\alpha^m)(x), \gamma_1(\beta^1) \circ \gamma_2(\beta^2) \circ \ldots \circ \gamma_m(\beta^m)(x)\big) \leqq$$

$$\leqq \varrho\big(\gamma_1(\alpha^1) \circ \gamma_2(\alpha^2) \circ \ldots \circ \gamma_m(\alpha^m)(x), \gamma_1(\alpha^1) \circ \gamma_2(\beta^2) \circ \ldots \circ \gamma_m(\beta^m)(x)\big) +$$

$$+ \varrho\big(\gamma_1(\alpha^1) \circ \gamma_2(\beta^2) \circ \ldots \circ \gamma_m(\beta^m)(x), \gamma_1(\beta^1) \circ \gamma_2(\beta^2) \circ \ldots \circ \gamma_m(\beta^m)(x)\big) \leqq$$

$$\leqq \varrho\big(\gamma_2(\alpha^2) \circ \ldots \circ \gamma_m(\alpha^m)(x), \gamma_2(\beta^2) \circ \ldots \circ \gamma_m(\beta^m)(x)\big) + K|\beta^1 - \alpha^1| \leqq \ldots$$

$$\ldots \leqq K \sum_{i=1}^{m} |\beta^i - \alpha^i| \leqq \sqrt{2m} K\, d(\bar{\varkappa}(a), \bar{\varkappa}(b)).$$

Therefore the assertion of the lemma holds with $\delta = \frac{1}{\sqrt{2m}K} \tilde{\delta}$.

The length of the tangent vectors of a differentiable manifold is usually defined after the introduction of a Riemannian metric. Here the length of tangent vectors of R will be defined at first to be the basic tool in establishing the required Riemannian metric. The velocity of orbits could be naturally considered as the length of their tangent vectors. The following lemma serves to prepare a general definition on this basis.

Lemma 3.3. *Let $\varphi: R^1 \to R$ be an orbit starting at $x \in R$ and $\varphi_*(0) \neq O_*$. If $\psi: [0, \alpha] \to R$ is a curve differentiable at 0 and $\psi(0) = x$, $\psi_*(0) = \lambda \cdot \varphi_*(0)$, $\lambda \geqq 0$, then $\lim\limits_{\tau \to 0} \frac{\varrho(\psi(\tau), \psi(0))}{\tau} = \lambda \varphi^*$.*

Proof. There is a canonical coordinate system of the second kind $\varkappa: U \to R^m$ at x with $\varkappa \circ \varphi(\tau) = (\tau, 0, \ldots, 0)$ for $\varphi(\tau) \in U$ by Lemma 2.4. Therefore $\lim\limits_{\tau \to 0} \frac{d(\varkappa \circ \varphi(\tau), \varkappa(x))}{\varrho(\varphi(\tau), \tau(x)} = \frac{1}{\varphi^*}$ and for a sufficiently small $\bar{\tau} > 0$ there is a $\tau > 0$ such that $\tau = d(\varkappa \circ \varphi(\tau), \varkappa(x)) = d(\varkappa \circ \psi(\bar{\tau}), \varkappa(x))$ and $\tau \to 0$ if $\bar{\tau} \to 0$. Hence

$$\left| \frac{\varrho(\varphi(\tau), x)}{\tau} - \frac{\varrho(\varphi(\tau), \psi(\bar{\tau}))}{\tau} \right| \leqq \frac{\tau(\psi(\bar{\tau}), x)}{\tau} \leqq \frac{\varrho(\varphi(\tau), x)}{\tau} + \frac{\varrho(\varphi(\tau), \psi(\bar{\tau}))}{\tau}.$$

If $\varkappa\circ\psi(\bar\tau)=(\psi^1(\bar\tau),\dots,\psi^m(\bar\tau))$ and $\bar\tau\to+0$ then

$$\limsup\frac{d(\varkappa\circ\varphi(\tau),\varkappa\circ\varphi(\bar\tau))}{\tau}=\limsup\left[2\left(1-\frac{\psi^1(\bar\tau)}{\tau}\right)\right]^{1/2}=$$

$$=\limsup\left[2\left(1-\frac{\psi^1(\bar\tau)}{d(\varkappa\circ\psi(\bar\tau),x(x))}\right]^{1/2}=0.$$

Therefore

$$\limsup\frac{\varrho(\varphi(\tau),\psi(\bar\tau))}{\tau}\leqq$$

$$\leqq\limsup\frac{\varrho(\varphi(\tau),\psi(\bar\tau))}{d(\varkappa\circ\varphi(\tau),\varkappa\circ\psi(\bar\tau))}\cdot\limsup\frac{d(\varkappa\circ\varphi(\tau),\varkappa\circ\psi(\bar\tau))}{\tau}=0$$

in consequence of the preceding lemma. This implies with respect to above inequalities that if $\tau\to+0$ then $\lim\frac{\varrho(\psi(\bar\tau),x)}{\tau}=\varphi^*$. But then $\lim\frac{\varrho(\psi(\bar\tau),x)}{\bar\tau}=$ $=\lim\frac{\varrho(\psi(\bar\tau),x)}{\tau}\cdot\lim\frac{d(\varkappa\circ\psi(\bar\tau),\varkappa(x))}{d(\varkappa\circ\varphi(\bar\tau),\varkappa(x))}=\varphi^*\cdot\lambda.$

Corollary. *If φ,ψ are orbits with $\varphi_*(0)=\psi_*(0)$ then $\varphi^*=\psi^*$.*

On account of the above corollary a function $F\colon TR\to R^1$ can be defined on the tangent bundle TR of R as follows: let $F(v)$ for a $v\in TR$ be the velocity φ^* of any orbit φ such that $\varphi_*(0)=v$. This function F will be called the *length of tangent vectors*. Obviously $F(v)=0$ if and only if $v=O_*$ for some $x\in R$ and in consequence of the preceding lemma F is positively homogeneous of order 1 on every tangent space of R. In order to show the continuity of F some preliminaries are needed. These are provided by

Lemma 3.4. *Let $\gamma_i\colon R^1\to G_0$ $(i=0,1,\dots)$ be 1-parameter groups with $\gamma_{0*}(0)=$ $=\lim_{i\to\infty}\gamma_{i*}(0)$. If φ_i is the orbit of γ_i starting at x_i and $x_0=\lim_{i\to\infty}x_i$ then $\varphi_0^*=\lim_{i\to\infty}\varphi_i^*$.*

Proof. It is suitable to consider the special case $x_i=x_0$ $(i=1,2,\dots)$ separately. Let $\varkappa\colon U\to R^m$ be a coordinate system of R at x_0 with $\varkappa(x_0)=(0,0,\dots,0)$. Since $\gamma_0(\tau)=\lim_{i\to\infty}\gamma_i(\tau)$ for $\tau\in R^1$ and $\Gamma\colon G_0\times R\to R$ is continuous $\varphi_0(\tau)=\lim_{i\to\infty}\varphi_i(\tau)$ for $\tau\in R^1$. Therefore there is a $\delta>0$ and a N such that $\varphi_i(\tau)\in U$ if $|\tau|<\delta$ and $i=0$ or $i\geqq N$. Since φ_i is differentiable $\varkappa\circ\varphi_i(\tau)=(a_i^1\tau+\tau\varepsilon_i^1(\tau),\dots,a_i^m\tau+\tau\varepsilon_i^m(\tau))$ if $|\tau|\leqq\delta$ and $i=0$ or $i\geqq N$, where $\varepsilon_i^l(\tau)=O(\tau)$, for $l=1,\dots,m$. Let $\eta_i(\tau)$, $i=0,1,\dots$, $\tau\in R^1$ be defined by $\varphi_i^*=\frac{\varrho(\varphi_i(\tau),\varphi_i(0))}{|\tau|}+\eta_i(\tau)$ and $\eta_i(0)=0$. In consequence of lemma 3.2, there is a K with $K\cdot d(\varkappa\circ\varphi_0(\tau),\varkappa\circ\varphi_i(\tau))\geqq\varrho(\varphi_0(\tau),\ \varphi_i(\tau))\geqq$

$\geqq |\tau|\cdot|\varphi_0^*-\varphi_i^*+\eta_i^*(\tau)-\eta_0(\tau)|$ if $|\tau|\leqq\delta$ and $i\geqq N$. Therefore $K\cdot|\tau|\left[\sum_{l=1}^{m}(a_0-a_i)^2\right]^{1/2}+$ $+K\cdot|\tau|\cdot\left[\sum_{l=1}^{m}(\varepsilon_0^l(\tau)-\varepsilon_i^l(\tau))^2\right]^{1/2}\geqq|\tau||\varphi_0^*-\varphi_i^*+\eta_i(\tau)-\eta_0(\tau)|$ and $\tau\to 0$ yields $K\left[\sum_{l=1}^{m}(a_0^l-a_i^l)^2\right]^{1/2}\geqq|\varphi_0^*-\varphi_i^*|$. But the coordinates of $\varphi_{i*}(0)$ are $(a_i^1,\dots,a_i^m)$ in the coordinate system $\varkappa$ and $\varphi_{0*}(0)=\lim_{i\to\infty}\varphi_{i*}(0)$ by the continuity of the differential $\Gamma_*\colon T(G_0\times R)\to TR$ of Γ. Therefore the assertion of the lemma follows in the special case.

In the general case to any preassigned $\vartheta>0$ there is a neighborhood V of x_0 with $|\gamma_0^*(x)-\gamma_0^*(x_0)|<\frac{\vartheta}{2}$ for $x\in V$ by Lemma 3. 1. Let $X_0\colon R\to TR$ be the Killing vector field corresponding to γ_0. In consequence of Lemma 2. 3 there is a normal field of canonical coordinate systems of the first kind $\varkappa_z\colon U_z\to R^m$ on a compact neighborhood U of x_0. Let $(\alpha_i^1,\dots,\alpha_i^m)$ respectively $(\bar\alpha_i^1,\dots,\bar\alpha_i^m)$ be the coordinates of $\varphi_{i*}(0)$ and $X_i(x_i)$ for $x_i\in U$ in the coordinate system $\varkappa_{x_0}\colon U_{x_0}\to R^m$. Since $\lim_{i\to\infty}\varphi_{i*}(0)=\varphi_{0*}(0)=X_0(x_0)=\lim_{i\to\infty}X_0(x_i)$, there is a neighborhood $V'\subset U_{x_0}$ of x_0 with $\left[\sum_{l=1}^{m}(\alpha_i^l-\bar\alpha_i^l)^2\right]^{1/2}\leqq\left[\sum_{l=1}^{m}(\alpha_i^l-\bar\alpha_i^l)^2\right]^{1/2}+\left[\sum_{l=1}^{m}(\alpha_0^l-\bar\alpha_0^l)\right]^{1/2}\leqq\frac{\vartheta}{2C^2K}$, where C is the upper bound given in Lemma 2. 3 and K is an upper bound guaranteed by Lemma 3. 2 for the coordinate system $\varkappa_{x_0}$. Let $(\xi_i^1,\dots,\xi_i^m)$ respectively $(\bar\xi_i^1,\dots,\bar\xi_i^m)$ be the coordinates of $\varphi_{i*}(0)$ and $X_0(x_i)$ in the coordinate system $\varkappa_{x_i}$ and K_{x_i} an upper bound given by Lemma 3. 2 for $\varkappa_{x_i}$ in case of $x_i\in U$. Then

$$|\varphi_i^*-\varphi_0^*| = |\gamma_i^*(x_i)-\gamma_0^*(x_0)| \leqq |\gamma_i^*(x_i)-\gamma_0^*(x_i)|+|\gamma_0^*(x_i)-\gamma_0^*(x_0)| \leqq$$

$$\leqq K_{x_i}\left[\sum_{l=1}^{m}(\xi_i^l-\bar\xi_i^l)^2\right]^{1/2}+\frac{\vartheta}{2}\leqq K_{x_i}\cdot\lambda(\varkappa_{x_i},\varkappa_{x_0})\left[\sum_{l=1}^{m}(\alpha_i^l-\bar\alpha_i^l)^2\right]^{1/2}+\frac{\vartheta}{2}\leqq$$

$$\leqq K\cdot\lambda^2(\varkappa_{x_i},\varkappa_{x_0})\frac{\vartheta}{2C^2K}+\frac{\vartheta}{2}\leqq\vartheta$$

if $x_i\in V\cap V'$.

Lemma 3. 5. *The function $F\colon TR\to R^1$ is continuous.*

Proof. Let $v_i\in T_{x_i}R$, $i=0,1,\dots$ be such that $v_0=\lim_{i\to\infty}v_i$. In order to prove $F(v_0)=\lim_{i\to\infty}F(v_i)$ it suffices, on account of the preceding lemma, to show the existence of 1-parameter groups γ_i such that if φ_i is the orbit of γ_i starting at x_i then $v_i=\varphi_{i*}(0)$ and $\gamma_{0*}(0)=\lim_{i\to\infty}\gamma_{i*}(0)$. Let $\psi_{0x_0}\colon R\to G_0/H_{0x_0}$ be the diffeomorphism defined at the introduction of the differentiable structure of R and $M\subset T_\varepsilon G_0$ a subspace such that $\Pi_{x_0}\circ\exp_\varepsilon\colon M\to G_0/H_{0x_0}$ is diffeomorphic on a neighborhood V of O_ε in M. Then a neighborhood V' of x_0 exists on which

$\Phi = (\exp_\varepsilon \circ (\Pi_{x_0} \circ \exp_\varepsilon)^{-1} \circ \psi_{0x_0})^{-1} : V' \to G_0$ is diffeomorphic. Put $\bar{x}_i = \Phi(x_i)$, $\bar{v}_i = \Phi_*(v_i)$ and $\tilde{v}_i = R_{\bar{x}_i^{-1}*}(\bar{v}_i)$ for $x_i \in V'$. Let γ_i be the 1-parameter group with $\gamma_{i*}(0) = \tilde{v}_i$ and φ_i the orbit of γ_i starting at x_i for i with $x_i \in V'$. Then $\bar{v}_0 = \lim_{i\to\infty} \bar{v}_i$ by the continuity of Φ_* and $\tilde{v}_0 = \lim_{i\to\infty} R_{\bar{x}_i^{-1}*}\ \bar{v}_i = \lim_{i\to\infty} \tilde{v}_i = v_0$ by the simultaneous continuity of $R_{\bar{x}*}$ in its argument and in $\bar{x}$. Hence $\gamma_{0*}(0) = \lim_{i\to\infty} \gamma_{i*}(0)$. But $x_i = \bar{x}_i(x_0)$ therefore $\varphi_i(\tau) = \gamma_i(\tau) \cdot \bar{x}_i(x_0)$ for sufficiently small $|\tau|$. Thus $\varphi_i = \Phi^{-1} \circ R_{\bar{x}_i} \circ \gamma_i$ and $\varphi_{i*}(0) = \Phi_*^{-1} \circ R_{\bar{x}_i*}(\gamma_{i*}(0)) = \Phi_*^{-1} R_{\bar{x}_i*}(\tilde{v}_i) = v_i$ if $x_i \in V'$.

What has been proved up to now concerning F can be summarized by stating that the differentiable manifold R with the length of tangent vectors F forms a C^1-Finsler manifold $[R, F]$. The induced metric space of $[R, F]$ can be defined as generally it is done in case of any C^1-Finsler manifold (See [2]) on the following way: If $\psi : [\alpha, \beta] \to R$ is a piecewise C^1-curve of R then $\mathcal{L}_F(\psi) = \int_\alpha^\beta F(\psi_*(\tau))\,d\tau$ is called the *F-length* of ψ. Let $\varrho_F(x, y)$ be the infimum of the F-length of piecewise C^1-curves joining $x, y \in R$, then ϱ_F is a distance function on R. The metric space $[R, \varrho_F]$ is called the *induced metric space* of $[R, F]$. In order to prove $[R, \varrho_F] = [R, \varrho]$ some preliminaries are needed. In what follows these are provided.

If $\psi : [\alpha, \beta] \to R$ is a continuous curve and it is rectifiable in the metric space $[R, \varrho]$ then its length $\mathcal{L}_\varrho(\psi)$ will be called its *ϱ-length*. The following lemma can be proved on essentially the same lines as an other one formulated for the case of symmetric manifolds (see [8]).

Lemma 3.6. *If $\psi : [\alpha, \beta] \to R$ is a piecewise C^1-curve of the differentiable manifold R then it is rectifiable in the metric space $[R, \varrho]$ and $\mathcal{L}_\varrho(\psi) = \mathcal{L}_F(\psi)$.*

Since the metric space $[R, \varrho]$ is finitely compact and convex this lemma has the following obvious consequence:

Lemma 3.7. *If $x, y \in R$ then $\varrho(x, y) \leqq \varrho_F(x, y)$.*

If the continuous curve $\psi : [\alpha, \beta] \to R$ is rectifiable in the metric space $[R, \varrho_F]$ then its length $\mathcal{L}_{\varrho_F}(\psi)$ is called its *ϱ_F-length*. In the case when ψ is a piecewise C^1-curve then evidently $\mathcal{L}_{\varrho_F}(\psi) \leqq \mathcal{L}_F(\psi)$, where according to a result of H. Busemann and W. Mayer (see [1], [2]) the equality holds for any piecewise C_1-curve ψ if and only if F has convex indicatrix in each tangent space $T_x R$ of R. But by Lemma 3.6 and 3.7 $\mathcal{L}_F(\psi) = \mathcal{L}_\varrho(\psi) \leqq \mathcal{L}_{\varrho_F}(\psi)$ for any such curve ψ. These imply

Lemma 3.8. *The function $F : TR \to R^1$ has convex indicatrix in every tangent space of R.*

The proof of the assertion that $\varrho_F(x, y) \leqq \varrho(x, y)$ for $x, y \in R$ requires some technicalities. These are given in the following

Lemma 3.9. *If $\varkappa: U \to R^m$ is a coordinate system of R at x and d the distance function of R^m then there is a neighborhood V of x and a K such that $d(\varkappa(a), \varkappa(b)) \leqq \leqq K\varrho(a, b)$ if $a, b \in V$.*

Proof. For the sake of an indirect argument let it be assumed that to any N and in arbitrary neighborhood of x there are points a, b with $d(\varkappa(a), \varkappa(b)) \geqq \geqq N\varrho(a, b)$. Let further $\varkappa_z: U_z \to R^m$ be a normal field of canonical coordinate systems on a neighborhood U' of x given according to Lemma 2. 3 and C the corresponding upper bound. Then

$$d(\varkappa(a), \varkappa(b)) \leqq \lambda(\varkappa, \varkappa_x) \cdot \lambda(\varkappa_x, \varkappa_a) d(\varkappa_a(a), \varkappa_a(b)) \leqq \lambda(\varkappa, \varkappa_x) \cdot C d(\varkappa_a(a), \varkappa_a(b))$$

for $a, b \in U \cap U'$. Let $\varphi: R^1 \to R$ be the fundamental orbit of the coordinate system $\varkappa_a$ passing through b and $\varphi(\beta) = b$ then $\dfrac{\varrho(\varphi(0), \varphi(\beta))}{\beta} \leqq \dfrac{C \cdot \lambda(\varkappa, \varkappa_x)}{N}$. Therefore a sequence φ_i, $i = 1, 2, \ldots$ of fundamental orbits of the coordinate systems of the above field can be given with $\lim\limits_{i \to \infty} \dfrac{\varrho(\varphi_i(\beta_i), \varphi_i(0))}{|\beta_i|} = 0$ where $\lim\limits_{i \to \infty} \beta_i = 0$. In consequence of Lemma 2. 3 there is no loss of generality by assuming the existence of a fundamental orbit φ_0 with $\varphi_0(\tau) = \lim\limits_{i \to \infty} \varphi_i(\tau)$, $\tau \in R^1$. Let $\eta_i(\tau)$ be defined by $\varphi_i^* = \dfrac{\varrho(\varphi_i(\tau), \varphi_i(0))}{|\tau|} + \eta_i(\tau)$ and $\eta_i(0) = 0$ for $i = 0, 1, \ldots$ and $\tau \in R^1$. If $\vartheta > 0$ is given then there is such a $\delta > 0$ that $\eta_0(\tau) \leqq \dfrac{\vartheta}{2}$ for $|\tau| \leqq \delta$ and a L with $|\eta_i(\delta) - \eta_0(\delta)| \leqq \dfrac{\vartheta}{2}$ for $i \geqq L$. But obviously $\eta_i(\tau)$ is decreasing for $\tau < 0$ and increasing for $\tau > 0$, therefore $\eta_i(\tau) \leqq \eta_i(\delta) \leqq |\eta_i(\delta) - \eta_0(\delta)| + \eta_0(\delta)$, if $|\tau| \leqq \delta$ and $i \geqq L$. Therefore in consequence of Lemma 2.3 and 3. 4 the equality $\varphi_0^* = \lim\limits_{i \to \infty} \varphi_i^* = 0$ holds in contradiction with the fact that φ_0 is a fundamental orbit.

Lemma 3.10. *If $x, y \in R$ then $\varrho(x, y) \geqq \varrho_F(x, y)$.*

Proof. It suffices to prove the inequality for the case when x, y and a metric segment joining them are in the coordinate neighborhood U of a coordinate system $\varkappa: U \to R^m$ and bounds δ, K of Lemma 3. 2 and 3. 9 exist for U. Let $\varphi: [\alpha, \beta] \to R$ be a segment of $[R, \varrho]$ with $\varphi(\alpha) = x$ and $\varphi(\beta) = y$. In consequence of the preceding lemma $\varkappa \circ \varphi: [\alpha, \beta] \to R^m$ is a rectifiable curve of R^m and therefore $F(\varphi_*(\tau)) = 1$ for almost every $\tau \in [\alpha, \beta]$ by Lemma 3. 3. Hence $\varrho(x, y) = \int\limits_\alpha^\beta F(\varphi_*(\tau)) d\tau$. Let a sequence of subdivisions of $[\alpha, \beta]$ be given by $\alpha = \tau_{0,i} < \tau_{1,i} < \cdots < \tau_{n_i - 1, i} < \tau_{n_i, i} = \beta$ $(i = 1, 2, \ldots)$, where the ith subdivision is a refinement of the $(i-1)$th with

$$\lim_{i \to \infty} \max \{\tau_{l,i} - \tau_{l-1,i}: l = 1, \ldots, n_i\} = 0$$

and $\varkappa\circ\varphi$ is differentiable at $\tau_{l,i}$ for $l=1,\dots,n_{i-1}$ $(i=1,2,\dots)$. If i is large enough then the coordinate polygon inscribed in $\varkappa\circ\varphi$ corresponding to the i th subdivision exists, i.e. there is a map $\psi_i:[\alpha,\beta]\to R^m$ where $\psi_i(\tau)=\varkappa\circ\varphi(\tau_{j,i})+$ $+\dfrac{\tau-\tau_{j,i}}{\tau_{j+1,i}-\tau_{j,i}}(\varkappa\circ\varphi(\tau_{j+1,i})-\varkappa\circ\varphi(\tau_{j,i}))$ for $\tau\in[\tau_{j,i},\tau_{j+1,i}]$, $j=0,1,\dots,n_i-1$. The F-length $\mathscr{L}_F(\psi_i)$ of ψ_i is $\sum\limits_{j=0}^{n_i-1}\int\limits_{\tau_{j,i}}^{\tau_{j+1,i}}F(\psi_{i*}(\tau))d\tau$. But obviously $\varphi_*(\tau)=\lim\limits_{i\to\infty}\psi_{i*}(\tau)$ if $\tau=\tau_{j,i}$ for some i,j and $\alpha<\tau<\beta$, therefore $F(\varphi_*(\tau))=1=\lim\limits_{i\to\infty}F(\psi_{i*}(\tau))$ for such τ by Lemma 3. 5. Let $f_i:[\alpha,\beta]\to R^1$ be defined by $f_i(\tau)=F(\psi_{i*}(\tau))$ for $\tau\in[\tau_{j,i},\tau_{j+1,i}]$, $j=0,1,\dots,n_i-1$ and sufficiently large i. Then $F(\varphi_*(\tau))=\lim\limits_{i\to\infty}f_i(\tau)$ for almost every $\tau\in[\alpha,\beta]$ and the functions f_i are uniformly bounded since

$$\begin{aligned}F(\psi_{i*}(\tau)) &= \lim_{\tau\to\tau_{j,i}+0}\frac{\varrho(\psi_i(\tau),\psi_i(\tau_{j,i}))}{\tau-\tau_{j,i}}\leqq\\ &\leqq \limsup_{\tau\to\tau_{j,i}+0}\frac{\varrho(\psi_i(\tau),\psi_i(\tau_{j,i}))}{d(\varkappa\circ\psi_i(\tau),\varkappa\circ\psi_i(\tau_{j,i}))}\cdot\limsup_{\tau\to\tau_{j,i}+0}\frac{d(\varkappa\circ\psi_i(\tau),\varkappa\circ\psi_i(\tau_{j,i}))}{\tau-\tau_{j,i}}=\\ &\leqq \limsup_{\tau\to\tau_{j,i}+0}\frac{\varrho(\psi_i(\tau),\psi_i(\tau_{j,i}))}{d(\varkappa\circ\psi_i(\tau),\varkappa\circ\psi_i(\tau_{j,i}))}\cdot\frac{d(\varkappa\circ\psi_i(\tau_{j+1,i}),\varkappa\circ\psi_i(\tau_{j,i}))}{\tau_{j+1,i}-\tau_{j,i}}\leqq\frac{1}{\delta}K,\end{aligned}$$

where $\delta>0$ and K are bounds given by Lemma 3. 2 and 3. 9. Therefore by Lebesgue's theorem $\varrho(x,y)=\lim\limits_{i\to\infty}\int\limits_\alpha^\beta f_i(\tau)d\tau$. But if a $\vartheta>0$ is given then

$$\left|\mathscr{L}_F(\psi_i)-\int\limits_\alpha^\beta f_i(\tau)\,d\tau\right|\leqq\sum_{j=0}^{n_i-1}\int\limits_{\tau_{j,i}}^{\tau_{j+1,i}}\left|F(\psi_{i*}(\tau))-F(\psi_{i*}(\tau_{j,i}))\right|d\tau<\vartheta$$

if i is large enough on account of Lemma 3. 5 and of the fact that the $F(\psi_{i*}(\tau_{j,i}))$ are uniformly bounded. Thus $\varrho(x,y)=\lim\limits_{i\to\infty}\mathscr{L}_F(\psi_i)\geqq\varrho_F(x,y)$.

The above lemma and its previous counterpart give

Lemma 3. 11. $[R,\varrho_F]=[R,\varrho]$.

The next step is to show that what F defines on R is actually a Riemannian metric. In proving this the following lemma is essential.

Lemma 3. 12. *If $v_1, v_2\in T_xR$ are linearly independent and $\varphi_1,\varphi_2:R^1\to R$ are orbits starting at x with $\varphi_{i*}(0)=v_i$, $i=1,2$ then* $\lim\limits_{\tau_1,\tau_2\to 0}\dfrac{\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))}{F(\tau_1v_1-\tau_2v_2)}=1.$

Proof. In fact this is a special case of a result of H. Busemann and W. Mayer in a changed form. To show this let $\varkappa:U\to R^m$ be a canonical coordinate system of the second kind at x with $\varkappa\circ\varphi_1(\tau_1)=(\tau_1,0,0,\dots,0)$ and $\varkappa\circ\varphi_2(\tau_2)=(0,\tau_2,0,\dots,0)$ for $\varphi_1(\tau_1),\varphi_2(\tau_2)\in U$. If $v\in T_zR$, $z\in U$ and $\varkappa(z)=(z^1,\dots,z^m)$, $v=(v^1,\dots,v^m)$ then $F(v)$ is given by $F_\varkappa(z^1,\dots,z^m;v^1,\dots,v^m)$ in the coordinate system $\varkappa$. Let $\psi:[0,1]\to U$

be defined by $\varkappa\circ\psi(\tau)=\varkappa\circ\varphi_2(\tau_2)+\tau(\varkappa\circ\varphi_1(\tau_1)-\varkappa\circ\varphi_2(\tau_2))$ for sufficiently small τ_1,τ_2, then $\psi_*(\tau)=(\tau_1,-\tau_2,0,\dots,0)$. Therefore

$$F(\tau_1v_1-\tau_2v_2)=F(0,\dots,0;\ \tau_1,-\tau_2,0,\dots,0)=$$

$$=\int_0^1 F_\varkappa(0,\dots,0;\ \tau_1,-\tau_2,0,\dots,0)d\tau=M(\psi)$$

which is a quantity introduced by H. BUSEMANN and W. MAYER, and according to their result $\dfrac{\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))}{M(\psi)}\to1$ if $\tau_1,\tau_2\to0$ (see [2]).

The length of tangent vectors $F:TR\to R^1$ defines a norm in each tangent space of R and the Finsler manifold $[R,F]$ is Riemannian if and only if all these norms are euclidean. Therefore to prove that $[R,F]$ is Riemannian it suffices to show that in the tangent spaces normed by F the metric angle of segments exist (see [7]). In doing this the same methods are used as applied by W. RINOW in analogous questions (see [7]).

Lemma 3.13. *In the tangent spaces T_xR of R normed by F the metric angle of segments exists.*

Proof. Let $v_1,v_2\in T_xR$ be linearly independent with $F(v_1)=F(v_2)=1$ and φ_1,φ_2 orbits starting at x with $\varphi_{i*}(0)=v_i$, $i=1,2$. Then

$$\omega(\tau_1,\tau_2)=\left|\cos\gamma(x;\varphi_1(\tau_1),\varphi_2(\tau_2))-\cos\gamma(0_x;\tau_1v_1,\tau_2v_2)\right|=$$
$$=\left|\frac{\varrho(x,\varphi_1(\tau_1))^2+\varrho(x,\varphi_2(\tau_2))^2-\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))^2}{2\varrho(x,\varphi_1(\tau_1))\cdot\varrho(x,\varphi_2(\tau_2))}-\frac{\tau_1^2+\tau_2^2-F(\tau_1v_1-\tau_2v_2)^2}{2\tau_1\tau_2}\right|.$$

If $\eta_i(\tau)$, $\tau\in R^1$, $i=1,2$ are the functions introduced in Lemma 3.4 then

$$\omega(\tau_1,\tau_2)=\left|\frac{(\tau_1+\tau_1\eta_1(\tau_1))^2+(\tau_2+\tau_2\eta_2(\tau_2))^2+\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))^2}{2(\tau_1+\tau_1\eta_1(\tau_1))\cdot(\tau_2+\tau_2\eta_2(\tau_2))}-\right.$$
$$\left.-\frac{\tau_1^2+\tau_2^2-F(\tau_1v_1-\tau_2v_2)^2}{2\tau_1\tau_2}\right|\leqq\left|\frac{\tau_1^2(1+\eta_1(\tau_1))^2+\tau_2^2(1+\eta_2(\tau_2))^2}{2\tau_1\tau_2(1+\eta_1(\tau_1))(1+\eta_2(\tau_2))}-\frac{\tau_1^2+\tau_2^2}{2\tau_1\tau_2}\right|+$$
$$+\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))^2\left|\frac{1}{2\tau_1\tau_2}-\frac{1}{2\tau_1\tau_2(1+\eta_1(\tau_1))(1+\eta_2(\tau_2))}\right|+$$
$$+\left|\frac{F(\tau_1v_1-\tau_2v_2)^2}{2\tau_1\tau_2}-\frac{\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))^2}{2\tau_1\tau_2}\right|\leqq\frac12\left\{\frac{\tau_1}{\tau_2}\left|\frac{1+\eta_1(\tau_1)}{1+\eta_2(\tau_2)}-1\right|+\frac{\tau_2}{\tau_1}\left|\frac{1+\eta_2(\tau_2)}{1+\eta_1(\tau_1)}-1\right|+\right.$$
$$+\left(\frac{\tau_1}{\tau_2}\frac{1+\eta_1(\tau_1)}{1+\eta_2(\tau_2)}+\frac{\tau_2}{\tau_1}\frac{1+\eta_2(\tau_2)}{1+\eta_1(\tau_1)}+2\right)\cdot(\eta_1(\tau_1)+\eta_2(\tau_2)+\eta_1(\tau_1)\eta_2(\tau_2))+$$
$$\left.+\left(\frac{\tau_1}{\tau_2}+\frac{\tau_2}{\tau_1}+2\right)\cdot\left(1+\frac{\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))}{F(\tau_1v_1-\tau_2v_2)}\right)\cdot\left|1-\frac{\varrho(\varphi_1(\tau_1),\varphi_2(\tau_2))}{F(\tau_1v_1-\tau_2v_2)}\right|\right\}.$$

Since the orbits have a metric angle $\lim_{\tau_1, \tau_2 \to 0} \cos \gamma(x; \varphi_1(\tau_1), \varphi_2(\tau_2))$ exists. In consequence of the preceding lemma and the above inequalities $\omega(\tau_1, \tau_2) \to 0$ if $\frac{\tau_1}{\tau_2}$ is fixed and $\tau_1, \tau_2 \to 0$. The function F as a norm defines a Minkowskian geometry in $T_x R$ therefore the limit of $\gamma(0_x; \tau_1 v_1, \tau_2 v_2)$ exists if $\frac{\tau_1}{\tau_2}$ is fixed and $\tau_1, \tau_2 \to 0$. These imply that $\lim_{\tau_1, \tau_2 \to 0} \gamma(0_x; \tau_1 v_1, \tau_2 v_2)$ exists.

What have been proved till now yield that $[R, F]$ is a C^1-Riemannian manifold. With respect to anomalies of such manifolds the following lemma is essential.

Lemma 3. 14. *The function $F: TR \to R^1$ defines a C^∞-Riemannian manifold on R.*

Proof. For $\alpha \in G_0$ let $\alpha_*: TR \to TR$ be its differential. If $v \in T_z R$ and $\alpha_*(v) = v'$ then there is an orbit φ starting at z with $\varphi_*(0) = v$. Since $\varphi' = \alpha \circ \varphi$ is a differentiable curve $\varphi'_*(0) = \alpha_*(\varphi_*(0)) = v'$. But then $F(v) = F(v')$ in consequence of the fact that α is a distance preserving transformation of $[R, \varrho]$ and of Lemma 3. 3. Therefore α is an isometric transformation of $[R, F]$. Let $v_1, \ldots, v_m \in T_x R$ be an orthonormal system and $\varkappa: U \to R^m$ a canonical coordinate system of the second kind at x defined by orbits $\varphi_1, \ldots, \varphi_m$ with $\varphi_{i*}(0) = v_i$, $i = 1, \ldots, m$ according to Lemma 2. 4. Therefore if $z \in U$ and $\varkappa(z) = (z^1, \ldots, z^m)$ then $z = \gamma_1(z^1) \circ \cdots \circ \gamma_m(z^m)(x)$ where γ_i is the 1-parameter group which defines φ_i, $i = 1, \ldots, m$. Let $g_{ij}(z^1, \ldots, z^m)$, $i, j = 1, \ldots, m$ be the components of the Riemannian tensor defined by F with respect to the coordinate system $\varkappa$ for $z \in U$. But $\gamma_{i*}(0)$, $i = 1, \ldots, m$ are linearly independent therefore 1-parameter groups $\gamma_{m+1}, \ldots, \gamma_n$ exist which define a canonical coordinate system of the second kind $\bar{\varkappa}: \bar{U} \to R^n$ of G_0 at ε. Thus $z'^i = \Gamma^i(\alpha^1, \ldots, \alpha^n; z^1, \ldots, z^m)$, $i = 1, \ldots, m$ if $\alpha \in \bar{U}$, $\bar{\varkappa}(\alpha) = (\alpha^1, \ldots, \alpha^n)$, $z \in U$, $\alpha(z) = z' \in U$. The functions Γ^i are C^∞ since $\Gamma: G_0 \times R \to R$ is a C^∞-map. In consequence of the special choice of the coordinate systems $u^i = \Gamma^i(u^1, \ldots, u^m, 0, \ldots, 0;\ 0, \ldots, 0)$, $i = 1, \ldots, m$ for $u \in U$. Since the elements of G_0 are isometric transformations

$$g_{ij}(0, \ldots, 0) = \delta_{ij} =$$

$$= \sum_{k,l=1}^{m} g_{kl}(u^1, \ldots, u^m) \frac{\partial \Gamma^k(u^1, \ldots, u^m, 0, \ldots, 0; 0, \ldots, 0)}{\partial z^i} \frac{\partial \Gamma_l(u^1, \ldots, u^m, 0, \ldots, 0; 0, \ldots, 0)}{\partial z^j}$$

for $i, j = 1, \ldots, m$, which considered as a system of equations for the $g_{kl}(u^1, \ldots, u^m)$, $k, l = 1, \ldots, m$ must have a unique solution. This together with the fact that the Γ^i are C^∞-functions yield that the g_{kl} are C^∞ as well, what obviously implies the assertion of the lemma.

It is to be noted that contrary to the circumstance that Lemmas 3. 1-12 do not assume the existence of the metric angle of orbits for the last one this is essential.

In fact Lemma 3. 14 cannot have an analogue in the case of Finsler manifolds as obvious examples of Minkowskian geometries show.

Results of this section are summed up in

Theorem 3. *Let $\Gamma: G_0 \times R \to R$ be a differentiable transformation group and the differentiable manifold R have a distance function ϱ such that the metric space $[R, \varrho]$ is finitely compact and convex. If the elements of G_0 are distance preserving transformations of $[R, \varrho]$ and the orbits of the 1-parameter groups of G_0 are rectifiable and have metric angle in $[R, \varrho]$ then there is a unique Riemannian metric on R such that its induced metric space is $[R, \varrho]$.*

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Abstract Spaces abd Approximation, Proceedings of the Conference held at the Mathematical Research Institute at Oberwolfach, July 18—27, 1968. Edited by **P. L. Butzer** and **B. Sz.-Nagy,** 423 pages, Birkhäuser Verlag, Basel—Stuttgart, 1969.

The Conference was dedicated to the memory of Jean Favard. The first paper, written by G. Alexits and M. Zamansky, gives an appreciation of Favard's life and results.

Most of the further forty papers deal with the theoretical aspects of approximation theory, but several papers treat the related fields of functional analysis and operator theory. The chapter headings will indicate the scope of the Conference in more detail. The papers are classified according to their subject matter into five chapters: *I. Operator Theory* (P. R. Halmos, I. I. Hirschman, Jr., R. G. Douglas, R. S. Phillips, B. Sz.-Nagy, U. Westphal), *II. Interpolation and Approximation on Banach Spaces* (G. G. Lorentz and T. Shimogaki, R. O'Neill, H. Berens, P. L. Butzer and K. Scherer, G. Alexits, I. Singer, B. Brosowski), *III. Harmonic Analysis and Approximation* (P. R. Masani, R. A. Hirschfeld, J.-P. Kahane, H. S. Shapiro, E. Görlich, G. Sunouchi, L. Leindler, J. L. B. Cooper, P. G. Rooney, T. K. Boehme), *IV. Algebraic and Complex Approximation* (T. J. Rivlin, R. B. Schnabl, M. W. Müller, P. O. Runck, M. V. Golitschek, E. Popoviciu, T. Popoviciu, J. Korevaar and C. K. Chui, P. C. Curtis, Jr.), *V. Numerical and Spline Approximation, Differential Equations* (A. M. Ostrowski, K. Zeller, J. Nitsche, A. Sharma and A. Meir, W. Walter, H. Günzler and S. Zaidman, J. Löfström).

In addition there is a report on new and unsolved problems based upon a special problem session and later communications from the participants. This part of the volume was edited by H. S. Shapiro.

The book is arranged very well and the printing is nice.

L. Leindler (Szeged)

Richard S. Palais, Foundations of global non-linear analysis, VIII+132 pages, New York—Amsterdam, W. A. Bejamin, Inc. 1968.

This paper is an expanded version of lectures held at the Mathematics Institutes of Bonn University and the University of Geneva during the summer of 1966.

It is the first systematic exposition and axiomatic foundation from a category theoretical standpoint of the subject of global non-linear analysis.

According to the author's conception, *concrete local linear analysis* is the study of the classical spaces of real, complex, or vector valued functions on R^n or on some domain in R^n, and of their linear maps (integro-differential operators). In *global linear analysis* the role of a domain in R^n is given to an arbitrary finite dimensional differentiable manifold M, while in *global non-linear analysis* the role of linear maps of function spaces on a differentiable manifold M is given to non-linear maps that are "locally approximable" by a linear map in some sense. As the invariant structure of the function spaces under these non-linear maps is an infinite dimensional differentiable manifold, the infinite dimensional manifolds and their differentiable maps are the subject of *abstract global non-linear analysis.*

These concepts are explained on concrete function spaces and maps (differential operators).

The author sets down and discusses some important problems in connection with these concepts, especially the index-problem of a non-linear elliptic differential operator and the generalized calculus of variations.

The reader is supposed to be familiar with the basic notions of the theories of categories and functors, of infinite dimensional differentiable manifolds, and of vector bundles, for example with Lang's "Introduction to Differentiable Manifolds" and with chapter IV in R. S. Palais' "Seminar on the Atiyah-Singer Index Theorem", Annals Study n. 57.

P. T. Nagy (Szeged)

S. Fenyő—T. Frey, Modern mathematical methods in technology, Vol. 1 (North-Holland Series in Applied Mathematics and Mechanics, Vol. 9), XII+407 pages, Amsterdam—London, North-Holland Publ. Co., 1969.

This is a translation, with some minor improvements, of the German original published in 1967 by Birkhäuser Verlag as vol. 8 in the International Series of Numerical Mathematics. The authors' aim is to acquaint the reader with mathematical disciplines important for recent applications. They had in mind in the first place natural scientists and engineers, but they also considered the mathematician interested in the latest fields of application.

Contents: 1. *Extension of the classical concept of an integral.* (Lebesgue integral. Stieltjes integral.) — 2. *The operational calculus.* (Concepts from algebra. The operational calculus of number sequences. The operational calculus of functions.) — 3. *Fundamentals of distribution theory.* (The distribution concept. Operations with distributions. Application to ordinary linear differential equations. Representation theorem. Distribution sequences. Fourier transformation of distributions. Regularisation of functions. Applications.) — 4. *Analysis of non-linear differential equations. The theory of nonlinear vibrations.* (Existence and uniqueness. Stability. Structure of the integral curve. Nonlinear vibrations.)

B. Sz.-Nagy (Szeged)

Herbert S. Wilf, Finite sections of some classical inequalities, IV+82 pages, Berlin—Heidelberg—New York, Springer-Verlag, 1970.

This book is very useful for those who are familiar with classical inequalities and intend to have a survey of new results connected with Hilbert's inequality and its generalizations, or with Toeplitz and Hankels forms.

If we consider any inequality of the form

$$f(x_1, x_2, \ldots) \leqq A \cdot g(x_1, x_2, \ldots) \qquad (g \geqq 0),$$

where the x_i are real variables and A is a best possible constant, then the best constants in

$$f(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \leqq A_n \cdot g(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$$

certainly satisfy $A_n \leqq A$. The object of this volume is to study refinements of the assertion $A_n \to A$ $(n \to \infty)$.

The book consists of four chapters. Chapter 1 contains some classical results including the inequality of Hilbert, Hardy, and Carleman, as well as the basic properties of Toeplitz forms. Chapter 2 discusses the theory of Toeplitz integral kernels, and Hilbert forms. Chapter 3 is concerned with Hankel forms and their spectral theory. The author discusses the boundedness of such forms, their lowest eigenvalues, and connections with orthogonal polynomials on a curve in the complex plane. Chapter 4 is devoted to inequalities which do not assert boundedness of some linear operator on l^2. Here much of the general theory is inapplicable and new methods have to be introduced to deal with special problems. The truncated version of Carleman's inequality is given to introduce the ideas, and extensions of the method to more general inequalities in l^p are presented.

The book contains reference to 55 items and a Subject Index. The author arranged his book lucidly and gave many useful hints to the literature; it is a valuable work.

L. Leindler (Szeged)

L. P. Hyrvärinen, Information Theory for Systems Engineers (Economics and Operations Research XVII), VIII+197 Seiten, Springer-Verlag, Berlin—Heidelberg—New York, 1970. — DM 44,—

Das Buch ist eine Ausarbeitung einer vom Verf. gehaltenen Vortragsreihe an dem IBM European Systems Research Institute in Genf.

Informationstheorie hat sich in dem letzten viertel Jahrhundert zu einer schönen, runden mathematischen Disziplin, aber gleichzeitig zu einem unentbehrlichen Hilfsmittel der Nachrichten- bzw. Computer-Technik entwickelt. Jeder, der heute ein Buch über dieses Gebiet schreiben will, soll sich sofort die schwierige, und oft den Erfolg des Buches entscheidende Frage stellen, ob er sein Werk für den mathematischen Feinschmecker, für den Praktiker oder evtl. für beide bestimmen will. Der Verf. des vorliegenden Buches ist dem Erbe C. E. Shannons, des Begründers der Informationstheorie, treu geblieben und hat, wie auch in dem Titel betont ist, ein für die Praxis bestimmtes, gut brauchbares Fachbuch geschafffen.

Entsprechend der Zielstellung des Buches werden nur soviel mathematische Grundkenntnisse vorausgesetzt, wie in jedem für Ingenieure gehaltenen Mathematik-Kurs enthalten sind, etwa die Grundlagen der Differential- and Integralrechnung und der Wahrscheinlichkeitstheorie. Es werden keine strengen mathematischen Definitionen, Sätze und Beweise angegeben, sondern die Begriffe und ihre Eigenschaften werden aus Plausibilitätsbetrachtungen hergeleitet. Die Darlegungen werden nicht nur soweit getrieben, bis die runde Theorie reicht, sondern bis um die Erfordernisse der Praxis. Der Praktiker wird sich besonders an die ausgearbeiteten Algorithmen (z. B. für Ermittlung eines optimalen Kodes) und an die schönen, besonders für die Computer-Technik bestimmten Anwendungen (Kapitel 5) freuen.

Das Buch, als eine Ausarbeitung einer Vortragsreihe, ist innerlich sehr zusammenhängend, die einzelnen Kapitel stützen wesentlich an die Vorangegangenen, dadurch eignet sich das Buch vorzüglich für diejenigen, die sich in die Informationstheorie einarbeiten wollen, aber es ist kein Nachschlagwerk. Die sebstständige Lösung der Aufgaben am Ende des Buches setzt den Leser imstande, während seiner Arbeit auftretende neuartige Probleme lösen zu können.

Die Kapiteltitel sind: 1. Einleitung; 2. Störungsfreie Kanäle; 3. Kodierung für störungsfreie Kanäle; 4. Gekoppelte Ereignisse, natürliche Sprachen; 5. Anwendungen der Kodierung ohne Störung; 6. Gestörte Kanäle; 7. Fehlerentdeckende und -korrigierende Kode; 8. Eigenschaften der Kanäle für stetige Signale; 9. Empfang stetiger Signale; 10. Informationsfilter. — Anhang über angewandte Hilfsmittel, Probleme und Lösungen, Literaturverzeichnis und Index machen den Band komplett.

D. Vermes (Szeged)

G. Takeuti—W. M. Zaring, Introduction to axiomatic set theory (Graduate Texts in Mathematics, Vol. 1), VII+250 pages, New York—Heidelberg—Berlin, Springer-Verlag, 1971. — DM 35

As the authors also state in the introduction, a systematic development is presented here of Zermelo—Fraenkel set theory, with a stress on detailed proofs rather than on the inclusion of a large number of deep results. The topics dealt with include the interrelation between the Generalized Continuum Hypothesis, the Aleph Hypothesis, and the Axiom of Choice. Gödel's model of constructible sets is presented, and Cohen's forcing method is developed in order to prove the indepence of the Axiom of Constructibility. Unfortunately, the use of formulas is slightly excessive in some parts of the book, while sufficient intuitive motivation is sometimes lacking. Also, J. R. Shoenfield's unramified approach to forcing is probably simpler than the presented one. Nevertheless, the book is a great help to anyone wanting to get acquainted with axiomatic set theory thoroughly and without (much) assistance from an instructor. The authors plan a further volume entitled *Axiomatic set theory* discussing, in a very general setting, relative constructibility, generalized forcing, and their interrelationship.

Attila Máté (Szeged)

J. K. Percus, Combinatorial methods (Applied Mathematical Sciences, 4), IX+194 pages, New York—Heidelberg—Berlin, Springer-Verlag, 1971. — US $ 6,50

In harmony with the aims of the series *Applied Mathematical Sciences*, the treatment of the subject is not too abstract, but nevertheless of good quality, so as to reach a wide circle of readers: students in mathematics, physics, and chemistry, etc. Combinatorics is a subject that especially yields to such a treatment. Some chapter headings might be cited to give an indication of the topics discussed: Set generating functions — Permutations with restricted position. The master theorem — Classification of partitions — Ramsey's theorem — Distribution of labeled objects — Random walk on lattices — The ballot problem — The dimer problem — Counting patterns on two dimensional lattices — The Ising model — Estimates of the Curie temperature — Spin correlations. As these headings show, some interesting applications are also dealt with in detail. The discussion is enlivened by the large number of elaborated examples, visibly separated from the core of the text.

Attila Máté (Szeged)

J. H. Wilkinson and C. Reinsch, Linear algebra (Handbook for automatic computation, Vol. II., Die Grundlehren der mathematische Wissenschaften in Einzeldarstellungen, Band 186), 4 figures, X+439 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1971. — Cloth US $ 20.80

Volume Ia of this series specified a restricted version of ALGOL to be employed throughout the Handbook, and volume Ib described its implementation on a computer. The subsequent volumes are to present algorithms in specific areas of numerical analysis. The present one consists of two parts: I. Linear systems, least squares and linear programming, and II. The algebraic eigenvalue problem. These are collections of contributions each of which deals with a numerical procedure, discussing its theoretical background, the range of applicability, and giving an ALGOL program. This can be fed into a computer without modification, or only a minor modification is needed, e. g. in order to make an economical use of the storage room by specifying when to use a backing store such as a magnetic tape. Most of the contributions had received pre-publication in *Numerische Mathematik*, and before being included in the present volume, the algorithms were thoroughly tested and, possibly, improved.

Attila Máté (Szeged)

R. von Mises — K. O. Friedrichs, Fluid Dynamics (Applied Mathematical Sciences, 5), IX+353 pages, 216 fig., Berlin—Heidelberg—New York, Springer-Verlag, 1971. — DM 24,—

From the Preface of the Editors (F. JOHN, J. P. LASALLE, L. SIROVICH): "In the summer 1941 Brown University undertook a Program of Advanced Instruction and Research in Mechanics." "Certainly an outstanding feature of this program must have been the lectures in Fluid Dynamics by Professor Friedrichs and the late Professor Mises. Their notes were prepared in mimeograph form and given a wide distribution at that time. Since their appearance these lectures have had a strong influence on teaching and research in the subject. — As the reader soon learns the notes have lost none of their vitality over the years. Indeed in certain instances only in the last few years has the field caught up with the ideas developed in the course of these lectures. Many ideas of value are still to be found in these notes."

Chapter headings: I. General theory of perfect fluids. II. Motion in two dimensions — Airwing of infinite span. III. Motion in three dimensions. IV. Theory of viscous fluids. V. Compressible fluids.

B. Sz.-Nagy (Szeged)

István Vincze. Mathematische Statistik mit industriellen Anwendungen, 440 Seiten, Budapest, Akadémiai Kiadó, 1971.

Dieses Buch ist die deutsche Fassung der ungarischen Originalausgabe (*Matematikai statisztika ipari alkalmazásokkal,* Budapest, Műszaki Kiadó, 1969), stimmt aber damit nicht vollständig überein; gewisse kleinere Änderungen wurden durchgeführt, weiterhin wurde der Text mit einem Paragraphen über den Rényi-Test und mit einem Abschnitt über das sequentielle Stichprobenverfahren ergänzt.

Nach einer wahrscheinlichkeitstheoretischen Einführung werden die Grundlagen der Stichprobenentnahme betrachtet: die verschiedenen Methoden der Stichprobenentnahme (einfache, zwei- und mehrstufige, sequentielle, geschichtete, Stichprobenentnahme mit Gruppierung), die Grundlagen der Theorie der geordneten Stichproben, Sätze von Glivenko, Kolmogoroff, Smirnow, Gnedenko und Koroljuk. Ein anderes Kapitel beschäftigt sich mit der Theorie der statistischen Schätzungen: die Begriffe der erwartungstreuen, konsistenten und stark konsistenten Schätzungen, die Wirksamkeit der Schätzungen, suffiziente Schätzungen, die Ungleichung von Cramér—Rao, Konstruktion von statistischen Schätzfunktionen, das Maximum—Likelihood—Prinzip, die Momentenmethode, Konfidenzintervalle für die wichtigsten Parameter einiger wichtigen Verteilungen, Konfidenzgürtel für die theoretische Verteilungsfunktion. Das folgende Kapitel enthält die Test-Theorie; die allgemeine Theorie der statistischen Tests, einfache und zusammengesetzte Hypothesen, parametrische und parameterfreie Probleme, die Wahrscheinlichkeit der Fehler erster und zweiter Art, Wählen des kritischen Bereiches, Vergleichung von Tests; parametrische Tests, μ-, F-, und t-Tests, Vergleich der Erwartungswerte zweier normalverteilter Zufallsveränderlichen im Falle von unbekannten und verschiedenen Standardabweichungen, Vergleich mehrerer Standardabweichungen, Testen der Normalität, Prüfen der Wahrscheinlichkeit eines Ereignisses; parameterfreie Tests, χ^2-Test, Anwendung zur Prüfung der Anpassung, der Homogenität und der Unabhängigkeit, Wilcoxon-Test, Kolmogoroff-Test, Rényi-Test. Ein sebständiges Kapitel beschäftigt sich mit der Varianzanalyse: Satz von Fischer—Cochran, einfache Klassifikation, zweifache Klassifikation, Tests ohne und mit Wechselwirkungen, dreifache Klassifikation, unvollständige Versuchsanordnungen, lateinisches Quadrat, zufällige Blöcke, ausgewogene unvollständige Blöcke, Kovarianzanalyse. Ein weiteres Kapitel wird der Korrelations- und Regressionsanalyse gewidmet: Methode der kleinsten Quadrate, Gleichung der Regressionsgeraden, Schätzung der Koeffizienten der Regressionsgeraden, Regressionsparabel, der Fall von mehreren Zufallsveränderlichen, Schätzung der Korrelationskoeffizienten, die wichtigsten statistischen Prüfungen, statistische Untersuchung der gemeinsamen Verteilung, Statistische Untersuchung des linearen funktionalen Zusammenhanges, Regression im Falle von Normalverteilung, Allokationsprobleme u. s. w. Ein Kapitel beschäftigt sich mit den statistischen Methoden der Qualitätskontrolle, was für die industriellen Anwendungen interessant ist, endlich folgt ein kurzes Kapitel über sequentielle Stichprobenverfahren. Am Ende des Buches findet man die wichtigsten Tafeln: Zufallszahlen zur Gleichverteilung, Zufallszahlen zur standardisierten Normalverteilung, Werte der Dichtefunktion der standardisierten Normalverteilung, Normalverteilung, Poisson-Verteilung, Binomialverteilung, F-, t- und χ^2-Verteilung, die kritischen Werte des Kolmogooff—Smirnowschen Zwei-Stichproben-Tests, des Wilcoxonschen Tests, des Vorzeichenstests und des Kolmogoroffschen Tests. Es gibt noch ein Schriftenverzeichnis, eine Liste über die wichtigsten Fachausdrücke in englischer und russischer Sprache, und ein Namen- und Sachregister.

Vorausgesetzt wird nur die Kenntnis der elementaren mathematischen Analysis. Das Buch ist in erster Reihe für diejenigen Leser gut brauchbar, die die statistische Anschauungsmethode und die wichtigsten statistischen Methoden kennenlernen und anwenden wollen.

Die Originalausgabe war die erste moderne Einführung in ungarischer Sprache in die mathematische Statistik und hat sich als Lehrbuch für Studenten sowie als Nachschlagwerk in der Praxis bewährt.

Wir wünschen der vorliegenden deutschen Version gleichen Erfolg.

K. Tandori (Szeged)

L. A. Lusternik and V. J. Sobolev, Elements of Functional Analysis (Authorized English translation, revised and corrected edition), X+322 pages, Hindustan Publishing Corporation (India), Delhi, 1971. — US $ 8.00

The original edition, in Russian, appeared in 1951; it was the first book on this area written in the Soviet Union. It grew out of an expository paper of the first author in the *Uspehi*, 1936, and of university lectures by the second author; a revised second edition in Russian appeared in 1965.

The first three chapters deal with the basic concepts of metric spaces, linear and normed linear spaces, and of linear functionals and operators. (The proof of the Hahn—Banach theorem is incomplete as it forgets about the limit ordinals.) Chapter 4 studies compact operators, and proves the Fredholm alternative if the underlying Banach space has a basis. Chapter 5 introduces to the spectral theory of bounded selfadjoint operator on Hilbert space (closely following parts of the "Spektraldarstellung" (1942) of the reviewer). The last chapter is on some problems of non-linear functional analysis (Fréchet differentials, implicit functions, tangent manifolds, etc.) — References to foreign authors are not always correct. J. von Neumann's name is never mentioned. Haar's name appears occasionally with the (wrong) adjective: "Viennese mathematician".

The book is a useful introduction to some aspects of functional analysis, although it would be better if it had followed the second edition in Russian, instead of the first.

Béla Sz.-Nagy (Szeged)

P. Deussen, Halbgruppen und Automaten (Heidelberger Taschenbücher, Band 99), 198 Seiten, Berlin—Heidelberg—New York, Springer Verlag, 1971. — DM 11,80,—

Dieses Taschenbuch ist einer reinen algebraischen Betrachtung der wichtigsten Ergebnisse der Automatentheorie gewidmet, so daß die Zustandsmenge eines Automaten als ein Rechts-Semimodul über die Eingangshalbgruppe aufgefaßt ist. Diese Auffassung ergibt eine Ähnlichkeit der

Automatentheorie mit der Theorie von Ringen und Moduln und inspiriert eine ganze Reihe nicht nur für die Automaten, sondern auch in der Theorie der Halbgruppen und Semimoduln interessanter Untersuchungen. Der aus dieser Auffassung stammenden Betrachtungsweise des Automaten widmet sich das vorliegende Buch.

Das erste Kapitel bringt die grundlegenden Definitionen und elementaren Sätze über Halbgruppen, die für den weiteren Verlauf notwendigen Ergebnisse über Idealtheorie von Halbgruppen, und einige Grundbegriffe über zweistellige Relationen und Kongruenzrelationen in Halbgruppen. Das zweite Kapitel ist der Theorie von Semimoduln gewidmet. Hier sind hauptsächlich die Homomorphismen, das direkte Produkt, die direkte Summe, das Tensorprodukt von Semimoduln und die Semimoduln mit Maximal- oder Minimalbedingung für Untersemimoduln untersucht. In diesem Teil findet man auch Ergebnisse über die streng zyklischen, vollreduziblen und irreduziblen Semimoduln und gewisse Resultate über die Konstruktionen der Kongruenzrelationen verschiedener Art. Daneben sind die Darstellungen von Semimoduln durch Graphen und die linearen Darstellungen behandelt. Das dritte Kapitel ist der Anwendung der in den ersten zwei Kapiteln aufgebauten Hilfsmittel in der Automatentheorie gewidmet. Nach der Einführung der wichtigsten automatentheoretischen Begriffen werden solche klassische Probleme, wie die Äquivalenz und Reduktion der Automaten, gelöst. Ferner sind interessante Sätze für den Fall, wenn die Eingangshalbgruppe des Automaten rechtskürzbar ist, bewiesen. Nach der Charakterisierung der durch Automaten induzierbaren Wortfunktionen wird der Leser mit einigen Fragen der Realisation der Automaten durch direkte Produkte und Superpositionen bekannt gemacht. Schließlich ist der bekannte Zusammenhang zwischen den Analysatoren und regulären Mengen im Falle, wenn die Eingangshalbgruppe des Automaten frei ist, entwickelt.

Trotz seinem nicht großen Umfang enthält das Buch ein reiches Material und gibt eine gute Übersicht über den Zusammenhang zwischen den Automaten und Halbgruppen. Die jedem Abschnitt beigegebenen Übungsaufgaben (insgesamt etwa 100 Übungen) dienen auch zur Ergänzung des Stoffes.

I. Babcsányi (Szombathely) — *I. Peák* (Szeged)

Robert Sauer, Differenzengeometrie, 234 Seiten mit 95 Abbildungen, Springer-Verlag, Berlin—Heidelberg—New York, 1970.

Bekanntlich versteht man unter Differenzengeometrie eine Behandlungsweise von differenzialgeometrischen Problemen, wobei Kurven bzw. Flächen zuerst durch Polygone bzw. Polyeder approximiert werden und dann die differenzialgeometrischen Sätze mittels eines Grenzüberganges aus elementargeometrischen Sätzen über Polygone bzw. Polyeder erhalten werden. Aus dieser Behandlungsweise ergeben sich zwei Vorteile, erstens ein geometrisch anschaulicher Einblick in die Tatsachen, die den differenzialgeometrischen Sätzen zugrunde liegen, zweitens die Möglichkeit der Anwendung von Approximationsmethoden, die bei Problemen der Technik nützlich sein können.

Dieses Buch gilt als die erste systematische Darlegung der Differenzengeometrie. Kapitel I bringt eine allgemeine Einführung, spezielle Flächen und insbesondere Probleme der Flächenverbiegung werden in Kapitel II behandelt. Die infinitesimalen Flächenverbiegungen werden in Kapitel III betrachtet, wobei sich auch projektiv-geometrische Beziehungen ergeben. Durch seine klare und geometrisch inhaltsreiche Darstellung bietet das Buch eine sehr lesbare Einführung in diese interessante differenzialgeometrische Theorie.

J. Szenthe (Szeged)

N. Bourbaki, Variétés différentielles et analytiques, Fascicule de résultats (Paragraphes 1 à 7, Éléments de mathématique, XXXIII, 97 pages, Hermann, Paris, 1967.

Ce fascicule présente les notions fondamentales et les principaux résultats de la théorie des variétés différentielles, sur le corps des nombres réels, et des variétés analytiques sur un corps valué complet non discret.

J. Szenthe (Szeged)

D. V. Anosov, Geodesic flows on closed Riemannian manifolds with negative curvature, Proceedings of the Steklov Institute of Mathematics, edited by I. G. Petrovskiĭ and S. M. Nikol'skiĭ, number 90 (1967). Translated from the Russian by S. Feder, III + 235 pages, American Mathematical Society, 1969.

The theory of geodesic flows on closed Riemannian manifolds of negative curvature is exposed here in the new approach of the author. His starting point is the observation that such flows satisfy a condition denoted by (U) which roughly speaking means that trajectories in the neighborhood of a fixed one behave like those close to a saddle. This condition is likewise formulated for systems with discrete time which are called cascades. Flows and cascades satisfying these conditions are called (U)-systems and in fact they form the subject matter here. The main results all due to the author are that a (U)-system is structurally stable in the sense of Andronov and Pontrjagin, and a technical one which surmounts those difficulties which occur in proofs of ergodicity in connection with changes of coordinates. These important results are given in a carefully detailed presentation presupposing a minimal familiarity with the basic concepts.

J. Szenthe (Szeged)

Leopold Fejér, Gesammelte Arbeiten. (Im Auftrag der Ungarischen Akademie der Wissenschaften herausgegeben und mit Kommentaren versehen von P. Turán, Mitglied der Ungarischen Akademie der Wissenschaften.) I—II, 872, bzw. 850 Seiten, Akadémiai Kiadó, Budapest, 1970.

Leopold Fejér war eine repräsentative Persönlichkeit der ungarischen Mathematik; seine Tätigkeit hat große Wirkung auf die mathematische Wissenschaft und auch auf das ganze mathematische Leben in Ungarn ausgeübt.

Die Ungarische Akademie der Wissenschaften hat ihrer edlen Pflicht damit genügt, daß sie die gesammelten Arbeiten von Leopold Fejér herausgegeben hat. Die schöne aber schwere Arbeit der Redaktion hat Paul Turán auf sich genommen. Nach dem Vorwort vom Redakteur und nach einer kurzen Lebensbeschreibung von Fejér (beide sind in zwei Sprachen, ungarisch und deutsch geschrieben) enthält dieses Werk die Arbeiten von Fejér in der Reihe ihrer Veröffentlichung. Nach den einzelnen Arbeiten gibt es Anmerkungen des Redakteurs; in diesen Anmerkungen werden die Wirkungen der entsprechenden Resultate skizziert.

Fejér hat seine Resultate meistens auch ungarisch publiziert; unter seinen Arbeiten gibt es welche, die nur ungarisch veröffentlich wurden. Die ungarischen und die fremdsprachigen Versionen zeigen manchmal gewisse Verschiedenheiten (besonders in den früheren Jahren). In den Fällen, wo die ungarische Version einer Arbeit kompletter als die andersprachige Version ist, gibt man hier die deutsche Übersetzung der ungarischen Version. Am Ende des zweiten Bandes gibt es einen Anhang, der diejenigen Resultate von Fejér enthält, die in Arbeiten von anderen Verfassern oder als Beispiele publiziert wurden. Leopold Fejér hat viele umfangsreiche Manuskripte nachgelassen; ihre Veröffentlichung bleibt eine zukünftige Aufgabe.

Károly Tandori (Szeged)

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